BULLETIN

OF THE

AMERICAN MATHEMATICAL SOCIETY

A Historical and Critical Review of Mathematical Science

EDITED BY

EARLE R. HEDRICK

DAVID R. CURTISS WILLIAM R. LONGLEY

WITH THE ASSISTANCE OF

HAROLD T. DAVIS HARRY W. KUHN
THEOPHIL H. HILDEBRANDT DERRICK N. LEHMER
JOHN R. KLINE HENRY L. RIETZ
CAROLINE E. SEELY

VOLUME XXXVI, NUMBER 7

JULY, 1930

PUBLISHED BY THE SOCIETY

MENASHA, WIS., AND NEW YORK

1930

PUBLISHED MONTHLY

Whole No. 370 $7.00 a Year

Entered as second class matter at the post office at Menasha, Wis., under the act of August 24, 1912.

PUBLICATIONS OF THIS SOCIETY

I. THE TRANSACTIONS. To members, $6.00; to others, $8.00.

Volumes 1-14, and 16-21: to members, $5.25; to others, $7.00.

Volumes 15, and 22-31: to members, $6.00; to others, $8.00.

II. THE BULLETIN. To non-members, $7.00.

Members of the Society receive the Bulletin if they have paid their dues, $5.00 of which is for subscription to the Bulletin.

Volumes 1-11 (and the three volumes of the Bulletin of the New York Mathematical Society) are out of print. The Society is glad to receive odd volumes or even odd numbers of these volumes.

Volumes 12-35: to members, $5.25; to others, $7.00.

III. THE AMERICAN JOURNAL OF MATHEMATICS.

Published by this Society and the Johns Hopkins University. Send orders to The Johns Hopkins Press.

To members, $5.75; to others, $7.50.

Complete sets (50 volumes): $350.00.

IV. THE COLLOQUIUM PUBLICATIONS.

Published in May, 1930:

Dunham Jackson, **The Theory of Approximation.** $2.50. (Volume XI of the Colloquium Series.)

For earlier issues, see another page.

Orders may be addressed to
450-458 AHNAIP ST., MENASHA, WIS., OR
501 WEST 116TH STREET, NEW YORK, N.Y.

On sale also by the following official agents of this Society:
The Open Court Publishing Co., 337 E. Chicago Ave., Chicago, Ill.
Bowes & Bowes, 1 Trinity St., Cambridge, England.
Hirschwaldsche Buchhandlung, Unter den Linden, 68, Berlin.

THE JUNE MEETING IN OREGON

The two hundred eighty-first regular meeting of the Society was held at the University of Oregon, Eugene, Oregon, on the morning of Friday, June 20, 1930. The meeting was called to order by President Hedrick, who was relieved during the session by Professors E. T. Bell, H. F. Blichfeldt, and R. E. Moritz. Professor E. E. DeCou was elected as acting Secretary in the absence of Associate Secretary T. M. Putnam.

The attendance included the following seventeen members:

E. B. Beauty, E. T. Bell, John Biggerstaff, H. F. Blichfeldt, Daniel Buchanan, L. G. Butler, J. W. Campbell, D. R. Davis, E. E. DeCou, F. L. Griffin, E. R. Hedrick, L. H. McFarlan, R. E. Moritz, J. M. Rankin, J. V. Uspensky, J. R. Vatnsdal, R. M. Winger.

A business meeting was held at the close of the morning session in which the following changes in the By-Laws were adopted as recommended by the Council of the Society at the April meeting.

(a) The first sentence of Article 10, Section 1, to read as follows:

The Society shall publish an official organ called the *Bulletin of the American Mathematical Society* which shall be sent to each member of the Society; five dollars of the dues paid by each member shall be for a year's subscription to this publication.

(b) The period at the end of the first sentence of Article 6, Section 2, to be replaced by a semi-colon (;) and to be followed by:

Five dollars of the dues of each member shall be for a year's subscription to the *Bulletin*.

It was voted to recommend to the Council that a meeting be held in June, 1931, at the University of Washington, it being understood that there will be a meeting at Thanksgiving at the University of California at Los Angeles, and a meeting in April at the University of California at Berkeley.

A luncheon for members and guests was provided at the dining-room of the University of Oregon, after which there were

automobile trips to the mountains and to the Oregon State College.

The titles of papers read at the meeting follow. Those whose abstract numbers are followed by the letter *t* were read by title. Mr. Small and Miss Swanson were introduced by Professor Griffin. The paper of Professor Milne was read by Professor DeCou.

1. *Residues of certain binomial coefficients for composite moduli*, by Professor E. T. Bell. (Abstract No. 36–5–278–*t*.)

2. *Unique decomposition*, by Professor E. T. Bell. (Abstract No. 36–5–279–*t*.)

3. *Rings whose elements are ideals*, by Professor E. T. Bell. (Abstract No. 36–5–280.)

4. *A theory of certain fields of points*, by Professor E. T. Bell. (Abstract No. 36–5–281–*t*.)

5. *An extension of the Alexander duality theorem*, by Dr. A. B. Brown. (Abstract No. 36–5–282–*t*.)

6. *Periodic orbits in the problem of three bodies with repulsive and attractive forces*, by Professor Daniel Buchanan. (Abstract No. 36–5–283.)

7. *Periodic orbits in the problem of four bodies with repulsive and attractive forces*, by Professor Daniel Buchanan. (Abstract No. 36–5–284.)

8. *The word "logarithm" used before the time of Napier*, by Professor Florian Cajori. (Abstract No. 36–5–285–*t*.)

9. *The cellular division and approximation of regular spreads*, by Dr. S. S. Cairns. (Abstract No. 36–7–290–*t*.)

10. *Note on ruled surfaces and their developables*, by Professor A. F. Carpenter. (Abstract No. 36–5–286.)

11. *On minimizing certain types of definite integrals*, by Professor D. R. Davis. (Abstract No. 36–7–291.)

12. *The expansion of analytic functions in the form* $b_0 + \sum_1^\infty b_n a_n z_n/(1-a_n z^n)$, by Dr. J. M. Feld. (Abstract No. 36–7–292–*t*.)

13. *Mutual inpedances of grounded circuits*, by Mr. R. M. Foster. (Abstract No. 36–5–287–*t*.)

14. *An application of a theorem of Junker*, by Professor Raymond Garver. (Abstract No. 36–5–288–*t*.)

15. *Varieties of polar singularities of non-analytic functions*, by Professor E. R. Hedrick. (Abstract No. 36–5–289–*t*.)

16. *Linear functional equations in the composite range of a function and n independent variables*, by Mr. L. S. Kennison. (Abstrat No. 36–7–293–*t*.)

17. *The problem of Lagrange of the calculus of variations for which the integrand contains the coordinates of the end points*, by Professor L. H. McFarlan. (Abstract No. 36–7–294.)

18. *Quadratic functional forms in a composite range*, by Professor A. D. Michal and Mr. L. S. Kennison. (Abstract No. 36–7–295–*t*.)

19. *On the maximum absolute value of the derivative of* $e^{-x^2}P_n(x)$, by Professor W. E. Milne. (Abstract No. 36–7–296.)

20. *On doubly-depleted Fourier's of the types* $\sum_1^\infty f(n) \cos(nx)$ *and* $\sum_1^\infty f(n) \sin(nx)$, *where* $f(n)$ *is a homogeneous function of* n, by Professor R. E. Moritz. (Abstract No. 36–7–297.)

21. *A center-surface for the ellipsoid*, by Mr. J. B. Small. (Abstract No. 36–7–298.)

22. *Curved wires of minimum attraction for a given particle*, by Miss Florence E. Swanson. (Abstract No. 36–7–299.)

23. *On the determination of stringing tensions for transmission lines and cables*, by Professor J. W. Campbell. (Abstract No. 36–7–300.)

24. *Incomplete numerical functions*, by Professor J. V. Uspensky. (Abstract No. 36–7–301.)

25. *Ch. Jordan's series for probability*, by Professor J. V. Uspensky. (Abstract No. 36–7–302–*t*.)

T. M. PUTNAM,
Associate Secretary

COBLE ON ALGEBRAIC GEOMETRY AND THETA FUNCTIONS

Algebraic Geometry and Theta Functions. By Arthur B. Coble. American Mathematical Society, Colloquium Publications, vol. X, New York, 1929. vii+ 282 pp.

The well known classical treatise by Krazer on the theory of θ functions contains several beautiful chapters dealing with the applications of this theory to algebraic geometry (in the largest sense of the word), but on the whole this treatise is more analytic than geometric in character. In it, page after page, swarms of complicated formulas and relations follow each other, rarely illuminated by a geometric interpretation. One can say, without fear of exaggeration, that all the geometric applications of this theory to algebraic curves and varieties made since Riemann and Weierstrass (Hurwitz, Poincaré, Schottky, Wirtinger, etc.) are absent in Krazer's treatise, or at most are only mentioned in short historical notes.

Having pointed out this gap in Krazer's treatment, we have thus indicated in advance the principal merit of Coble's new volume: the geometric spirit, which seeks to discover the geometric reality hidden behind the formal properties of the functions θ. The picture of this geometric reality, which the author draws before our eyes, is as varied as it is colorful: applications to the geometry on algebraic curves and surfaces; connections, as elegant as unexpected, with the Cremona transformations of the plane and of the space; analogies with the finite modular geometry and group theory aspect of the different geometric problems. It is true that the author, an expert algebraist true to the best traditions of the symbolic calculus of Clebsch-Aronhold, utilizes here and there a formidable algebraic apparatus, where the geometer would have preferred a simplified synthetic treatment. But this method has the positive advantage of facilitating the reading for those who see geometry best through algebraic glasses.

The main subject of this volume is the study of the canonical sextic C^6 of genus 4 in the space S_3, and in particular the determination of its tritangent planes. The author makes use of Wirtinger's representation of the C^6 as the locus W of the diagonal points of the quadruples of points of a g_4^1 on a general quartic curve. The locus W, the noted sextic of Wirtinger-Caporali, has 6 nodes at the vertices of a complete quadrangle. If W is given, then one of the $2^6-1=255$ systems of the contact quadrics of C^6 and all of the 28 tritangent planes contained in the system are rationally known. By means of the linear system of the ∞^2 adjoint cubics of W the plane is mapped upon one of the 255 Cayley cubic surfaces on C^6.

The consideration of Wirtinger's sextic does not lead, however, to the determination of all the 120 tritangent planes of C^6. It is here that Coble's analysis connects with the investigations by Schottky, relative to the modular functions of genus $p=3,4$. Schottky has shown that the coordinates of the 8 base points of a net of quadrics in S_3 can be expressed in terms of the zero values of the functions θ of genus 3 (modular function of genus 3). Similarly,

taking as starting point the θ-relations in the case $p=4$, Schottky defines a special set of 10 points in S_2, the geometric peculiarity of which lies in the fact that these points can be taken as the 10 nodes of a surface of the 4th order, known as Cayley's symmetroid. The author investigates very closely the two sets, denoted by P_8^3 and P_{10}^3 respectively, and shows that they enjoy, in their respective fields, very similar properties both with respect to the transformations of the moduli and to Cremona transformations. In the case $p=3$ the set P_8^3 gives the complete solution of the problem of the determination of the 28 double tangents of the general plane quartic of genus 3. Shall we expect the set P_{10}^3 to play a similar part in the case $p=4$, as far as the problem of the tritangent planes of the C^6 is concerned? With this question, left unanswered, the author's investigation of that particular problem terminates, although the author declares himself strongly in favor of the belief that a suitable geometric construction in S_2, which should associate with the set P_{10}^3 a curve of genus 4, must not fail to bring forth the connection between Cayley's symmetroid and the problem of the tritangent planes of C^6.

Without discussing the conjecture of the author, which seems to us very ingenious indeed, we want to make the following remark. In the case $p=3$ the determination of the double tangents depends on a geometric construction in the space S_3, and not in the space S_2 of the relative canonical curve. Would it not be as logical to expect that in the case $p=4$, the space S_4 and not S_2 should be the natural site of the desired geometric configuration? Apart from conjectures, the problem formulated by the author is a most interesting one, especially in view of its possible generalization to any value of the genus p.

The author treats very completely the case of a special C^6, characterized by the condition of lying on a quadric cone, or, what is the same, by the condition that the two g_3^1's, which on a general C^6 are distinct and are each the residual of the canonical series with respect to the other, coincide into one self-residual g_3^1. The equivalent transcendental condition is $\theta((0))=0$ (Weber). This C^6 is birationally equivalent to the locus of the invariant points of a Bertini involution, which locus is a curve of order 9 with 8 triple points at the fundamental points of the involution. The 120 degenerate sextics with nodes at the fundamental points give all the proper solutions of the problem. Schottky has given a direct definition of the set P_6^2 in terms of the modular functions. Coble's investigation completes this formal definition from a geometric point of view.

The questions outlined above, although of evident geometric interest, are of a special character. But in the treatment of these questions the author finds an opportunity to establish very general and important connections between Cremona transformations on one side and the theory of θ-functions on the other. This parallelism between the two fields constitutes perhaps the most original part of the author's work. There is first and above all the concept of *congruent* sets of points in the plane or in space. Two ordered sets of m points each, P_m and Q_m, are said to be congruent if there exists a Cremona transformation T with $\rho \leqq m$ F-points, such that the F-points of T are in P_m, the F-points of T^{-1} are in Q_m, and the $m-\rho$ remaining points of the two sets are corresponding points in T. The number of projectively distinct sets congruent to a given set P_m is in general infinite, but becomes finite for projec-

tively particular sets and also, in the plane, when $m<9$. The study of some of the most important examples of these particular sets P_m constitutes the foundation upon which the most essential part of the investigation is based, and it is here that the correlation between Schottky's investigations and the author's own results is most expressive. This is due to the fact that, according to Schottky, the coordinates of the points of these sets can be expressed in terms of the modular functions. Now, Coble establishes the beautiful theorem that the sets of points thus defined are carried by period transformations into congruent sets of points. This theorem is in correlation with the following general principle:

Let $g_{m,k}$ be the group of linear integral transformations, associated with the different types of Cremona transformations of S_k with $\rho \leqq m$ F-points (Kantor). If the coefficients of the transformations of $g_{m,k}$ are reduced mod 2, a finite group is obtained which is isomorphic to the group of the half-period transformations for a suitable value of p, or to a subgroup of this group.

The author develops a very elegant and important representation of the group $g_{m,k}$ as an infinite (discontinuous) group $G_{m,k}$ of Cremona transformations in a space $S_{m(m-k-2)}$, by mapping projectively distinct sets P_m of S_k upon the points of the said space. Each transformation of $G_{m,k}$ represents a type of Cremona transformation of S_k with $\rho \leqq m$ F-points, and corresponding points of the transformation in $S_{k(m-k-2)}$ arise from congruent sets in S_k. The group $G_{m,k}$ describes, so to speak, the purely discontinuous parameters on which the Cremona transformations of S_k depend.

Another excellent aspect of the volume is the use of the finite modular geometry in the exposition of the theory of characteristics. The representation of the half-period characteristics and of the θ-characteristics by means of points and quadrics of a modular space S_{2p-1} has a great descriptive and didactic value. The elements of the theory of characteristics, which form the content of the second part of Krazer's treatise, are presented in a very simple and suggestive form, thanks to the use of the language of finite geometry.

Given the subject and the nature of the problems treated, the book is not exactly, and one could not expect it to be, easy reading. Perhaps the abundance of details does not make the reading easier. However, the general lines of the exposition are well planned and the style is clear and concise. We feel that, thanks to its rich geometric content and originality of treatment, Coble's work is a really important contribution to the theory and application of the θ-functions, and will be well accepted by competent readers.

Table of Contents:

Chapter I. Linear systems of curves and Cremona transformations. Birational transformations and the geometry on an algebraic curve.
Chapter II. The elements of the theory of θ-functions. Finite geometry and θ-functions. θ relations. The generalized variety of Kummer.
Chapter III. Geometric applications of the modular functions of genus 2.
Chapter IV. Geometric applications of the modular functions of genus 3.
Chapter V. Geometric aspects of the modular abelian functions of genus 4.
Chapter VI. θ-relations of genus 4.

OSCAR ZARISKI

DICKSON'S THEORY OF NUMBERS

Introduction to the Theory of Numbers. By Leonard Eugene Dickson. The University of Chicago Press, 1929. vi+183 pp.

The appearance in English of a thoroughly sound and scholarly introduction to the theory of numbers, illuminated by many striking flashes of originality in both substance and presentation, is no commonplace in the progress of mathematics. For while it is true, as the author says, that "during twenty centuries the theory of numbers has been a favorite subject of research by leading mathematicians and thousands of amateurs," the theory has had fewer books, good, bad or mediocre, written around it than any other of the recognized major fields of mathematics. In English there are probably less than six treatises devoted entirely to the subject, and these range from the uncompleted work of Mathews to brief descriptions of the rudiments of the theory, quite inadequate to give a reader a working knowledge of even the meagre field covered. Although the merits of some of these older books have long been recognized, they belong to a past age, and are in fact hopelessly oldfashioned. The like holds also in a lesser degree for some of the great European treatises. New works will not destroy the inherent values of the old, but the need for something livelier than a lukewarm rehash of the imperishable *Disquisitiones Arithmeticae* of Gauss as an introduction to the theory of numbers is badly needed, and this Professor Dickson has supplied.

With a uniquely comprehensive and detailed grasp of the history of the subject, the author is in a position to select from the bewildering mass of material available those parts of highest interest in which more than a mere semblance of vitality still resides. So far as is possible with the means to which he necessarily has restricted himself in an introduction, namely nothing more advanced than college algebra as understood in America, the author has presented a living picture of the classic theory as left by Gauss and of more than one outstanding advance since Gauss wrote his masterpiece 130 years ago. For example, Thue's great theorem and its generalizations (1909–1921) are here presented in English for the first time, and with extremely simple proofs. We shall recur to this later.

Again, as a long overdue deviation from the Gaussian tradition, Dickson has used his unrivalled historical knowledge to restore to their proper position certain fruitful concepts which have been neglected for over a century. Of these may be noted in particular the idoneal numbers of Euler, the return to general rational integer coefficients in binary quadratic forms as advocated by Kronecker, and the restoration of pre-Gaussian diophantine analysis, the origin of the whole vast theory of forms, to a place of prominence. With this restoration of the old in modern shape, arithmeticians will have but slight excuse for declaring that certain problems are no longer of importance merely because all methods, ancient and modern, have thus far failed to exhibit solutions.

Some idea of the scope of this work can be gained from the following paraphrase of its contents: Chapter I, Divisibility; II, Theory of Congruences;

III, Quadratic Reciprocity; IV, Introduction to Diophantine Equations; V, Binary Quadratic Forms; VI, Certain Diophantine Equations; VII, Indefinite Binary Quadratic Forms; VIII, Solution of $ax^2+by^2+cz^2=0$ in Integers; IX, Composition and Genera of Binary Quadratic Forms; X, The Thue-Siegel Theorems; XI, Minima of Real, Indefinite Binary Quadratic Forms.

Many matters in these concise, beautifully clear chapters would call for more extended comment in a complete notice; here only a few can be selected to give the flavor of the whole to different tastes.

Progress is rapid. By page 5, for instance, a reader will have sufficiently grasped the elementary properties of primes to be able to prove readily that there are infinitely many primes $4n-1$, as stated in example 12. On page 9, example 11, the converse of Fermat's theorem is stated with a hint of the proof. By page 14, the number of roots of a congruence has been disposed of, and by page 20, the essential theory of primitive roots is out of the way. On page 21 new ground is broken in the theory of residual polynomials and congruences. Here Dickson presents, among other things, an elegant reworking of the inclusive theory of Kempner (1921) in about five pages. The exercises on page 28 contain several general theorems of interest. In Chapter III, 8 pages (small ones at that), quadratic residues, including the reciprocity laws for the symbols of Legendre and Jacobi, are treated with all sufficient detail. The third proof of Gauss is given, also Eisenstein's geometrical equivalent of the second part.

In Chapter IV there is a refreshing departure from the tradition of most writers of treatises on the theory of numbers: diophantine analysis is introduced in a simple but severely scientific dress. Most of the general theorems in this chapter are from the author's original work of the past few years, although this is not stated in the text. The good old times of diophantine analysis, when writers were content with *some* solutions of their problems, are gone, let us hope, forever. Just as no decent analyst today is content with anything but the best conditions under which his theorems hold, so the devotee of the theory of numbers demands *all* solutions or *none*. This is a hard ideal to attain, and exceptional circumstances may justify the publication of partial results. Dickson however has set himself the whole problem or nothing, and he solves it for such things as *all* integral solutions (x, y, z) of $ax^2+bxy+cy^2=ez^2$ where a, b, c, e are integers, $e\neq 0$, b^2-4ac not square, in terms of any given solution (ξ, η, ζ). As an exercise on page 49, the reader is asked to apply the general principles developed to all rational solutions of $aX^2+bY^2+cZ^2=e$, given one solution, and to observe the radical difficulty in attempting to pass thence to all integral solutions. Included in this chapter, for the first time apparently in a treatise, are the curious and interesting theorems on sums of integers having equal sums of like powers. Among other gems is an extremely simple derivation of all rational solutions of $W^2+X^2+Y^2+Z^2=0$. It is pointed out that the complete solution of $X^4+Y^4=Z^4+W^4$ in integers is yet to be found.

In Chapter V, binary quadratic forms $ax^2+bxy+cy^2$ are fully treated with respect to equivalence, reduction, number of representations, and genera, all in 28 pages. Following custom, the theory is carried first as far as possible until the separation into definite and indefinite forms becomes inevitable. As in the rest of the book, this subject, although long classical, is presented

with remarkable freshness and, a gift beyond all price to successive generations of students who will not henceforth be compelled to hack their blind path through scores of pages of unnecessary details, astonishing brevity and equally astonishing clarity. To mention only two novel details, which will be of great service to those working on numbers of representations in n-ary quadratic forms ($n>2$), we may cite the table on page 85 of reduced, positive, primitive forms of discriminant $-\Delta$ with a single class in each genus, for $\Delta<400$, and the statement on page 89 of those Δ, where $400<\Delta<23000$, for which there is a single class of positive, primitive forms of odd discriminant $-\Delta$. In the same direction, the results of many of the exercises in this chapter are of great value as savers of labor.

In Chapter VI, diophantine equations again appear, this time first in a thoroughgoing investigation of all integral solutions of $x^2-my^2=zw$. The discussion is a model of what modern diophantine analysis should be and may become under competent hands. To discuss $ax^2+bxy+cy^2=zw$, a slight levy is made on the resources of the preceding chapter. The method of Euler and Lagrange (forms that repeat under multiplication) is glanced at and discarded on account of its lack of generality. In this connection, on the algebraic side, the author might well have cited his own fundamental papers. Arithmetical generality, it would seem, can be attained in some instances at least by the machinery of ideals. But, even in these instances, it remains to be shown that the problem is actually solved in a constructible manner and not merely disguised by a restatement in sophisticated and artificially simpler language.

Indefinite binary quadratic forms receive in Chapter VII (18 pages) an adequate treatment equally as brief and as clear as that of positive forms in V. The reviewer does not recall another presentation of this classical theory, which in spite of its general beauty is rather messily detailed in spots, approaching this one in economy of space and effort.

The famous equation $ax^2+by^2+cz^2=0$ of Legendre and Gauss is the subject of Chapter VIII. Dickson essentially follows Dedekind's proof of the criteria for non-trivial solutions. The proof requires only four pages. Modern work by Ramanujan and Dickson and his students on universal forms (those representing all integers) and regular forms (universals except as to integers in certain arithmetic progressions), has demanded a closer scrutiny of certain aspects of ternary quadratic forms than has been customary in the literature. Together with the classical theory, the ground for these newer ideas is carefully prepared in this chapter.

Composition and the related problems of genera, which can be made quite tedious if clumsily handled, are elegantly treated in Chapter IX. What will probably strike a practised eye at once in this presentation is the complete absence of long formulas. Dedekind's proof of Gauss' theorem on duplication is given in less than two pages, both clear.

A new topic, whose difficulties have taxed experts in the subject, is discussed in the five pages of Chapter XI. Here it is a question of finding the lower bound of the absolute values of the numbers represented by any real indefinite binary quadratic form of given discriminant. The locus classicus here is Markoff's paper of 1879, which extends the main result to an infinitude of forms each having a minimum exceeding a certain prescribed limit. Dickson

produces short proofs of the key theorems, which are simpler than any others in the literature. An exposition of Markoff's extension is promised in the author's forthcoming sequel to this *Introduction*. The whole subject of minima has taken on a new importance in the light of modern methods.

To return to Chapters IX, X, in some respects the crown of the work. Outside of the brilliant achievement in the modern analytic theory of numbers at the hands of Hardy, Landau, Littlewood and their followers, arithmetic since Gauss has few results to offer that are so completely general and altogether satisfying as the theorem (1909) of Axel Thue. It is not a theory, it is a theorem. Some may prefer the abstract beauty of the theory of ideals, overlooking the fact that so far it has failed to solve the problem for which it was first created. The appeal of Thue's theorem is of a totally different order: a general problem had been outstanding for centuries; Thue devised a weapon of extraordinary power for attacking special cases of the problem and, in a significant sense, gave a solution of the general problem. Its entire simplicity marks Thue's theorem as one of the universal, natural things which are the very essence of the theory of numbers.

Here it is, as stated in Dickson's *Introduction*.

THEOREM 107. Let $f(z)=a_nz^n+\cdots+a_0$ be an irreducible polynomial of degree $n\geqq 3$ with integral coefficients. Consider the corresponding homogeneous polynomial $H(x, y)=a_nx^n+a_{n-1}x^{n-1}y+\cdots+a_1xy^{n-1}+a_0y^n$. If c is an integer, $H(x, y)=c$ has either no solution or only a finite number of solutions in integers.

THEOREM 116. Theorem 107 holds also if we omit the assumption that $f(z)=0$ is irreducible, but assume that all its roots are distinct, and $c\neq 0$.

Siegel's generalization (Theorem 113) provides a generalization of 107 different from 116, concerning $H(x, y)=G(x, y)$, where $G(x, y)$ is another polynomial (not perfectly general).

Thue's own proof of his great theorem is elementary and hard (a lacuan was first supplied by Maillet). Landau's presentation (following largely Siegel) in his classic *Vorlesungen* appeals to algebraic numbers and Farey fractionss and it demands considerable sophistication on the part of the reader. Dickson', signal service has been the reduction of the proof to a strictly elementary form. The word "elementary" is here meant in the sense of college algebra.

In closing this notice of an outstanding contribution to modern mathematical literature, we may emphasize again its clearness, its width of vision, and the unerring precision with which it consistently "hews to the line." The problems have been mentioned more than once. They are approachable and, in the opinion of the reviewer, afford the one decent set of exercises in existence on the, theory of numbers. The first volume of a sequel to the present *Introduction* containing much original work of the author and his students, is already in press. It is safe to predict that readers will find in it a worthy successor to the present. Dickson's work is above all else modern, independent in outlook, and thoroughly alive. Perhaps he will give us a book on the theory of algebraic numbers after the present series is completed. For the analytic theory we have Landau, but nothing comparable to it in English.

Siegel, I believe (if I am mistaken, I apologize), has said that the next great step in mathematical progress will be the burning of all books on mathematics. Should that somewhat spectacular step be taken, let us hope that Dickson's *Introduction* and a few other works of mathematical art escape.

E. T. BELL

THE NEW MECHANICS

Einführung in die Wellenmechanik. By Louis de Broglie. Translated into German by Rudolf Peierls. Leipzig, Akademische Verlagsgesellschaft M. B. H., 1929. iv+221 pp.

La Nouvelle Mécanique des Quanta. By George Birtwistle. Translated into French (and augmented by four appendices) by M. Ponte and Y. Rocard, with a preface by J. Hadamard. Paris, Albert Blanchard, 1929. vi+333 pp.

Gruppentheorie und Quantenmechanik. By Hermann Weyl. Leipzig, S. Hirzel, 1928. viii+288 pp.

About two years ago the new wave mechanics acquired remarkable experimental support through the discovery of the hitherto quite unknown phenomenon of diffraction of electronic streams by crystals. This discovery affords a genuine counterpart to the earlier discovery of the photoelectric effect, for the first shows that heretofore one aspect of the nature of matter has been left out of consideration while from the second it appears that one aspect of the nature of light had long been overlooked. It is now generally recognized that the behavior of light is corpuscular as well as undulatory; the diffraction of an electronic stream compels us to recognize that the behavior of matter is undulatory as well as corpuscular.

The older quantum theory failed to meet the dilemma of this two-fold character of light and of course did not consider at all the corresponding duality in the nature of matter; and it found no explanation of the presence of half quantum numbers in the formulas of the Zeeman effect and band spectra. A new theory was therefore inevitable. The first step was taken by Louis de Broglie in his dissertation (1924). The wave mechanics initiated by de Broglie received remarkable development at the hands of E. Schrödinger. Another line of development is due to W. Heisenberg.

De Broglie's *Wellenmechanik* furnishes the best introduction to the new quantum mechanics which has come to the reviewer's attention. A reader who desires to begin with a more elementary exposition would probably do well to use the second edition of *Materiewellen und Quantenmechanik* by A. Haas (1929); in this book the simpler aspects of the new theories are presented in an illuminating way. The exposition by de Broglie would then serve to complete an introduction to these most remarkable developments of the newer physics.

The starting point of de Broglie's wave mechanics has its origin in his purpose to develop the theory in such a way that there shall be an intimate connection between the conception of a corpuscle and that of periodicity in

phenomena so that the notion of the movement of the corpuscle shall be inseparable from that of wave propagation. How this intimate connection is brought about is best illustrated by the case of a single particle.

In order to present this example let us start from the Lorentz transformation of the restricted theory of relativity, taking this transformation in the form

$$x_0=x,\quad y_0=y,\quad z_0=(z-vt)/(1-\beta^2)^{1/2},\quad t_0=(t-\beta t/c)/(1-\beta^2)^{1/2},$$

where the coordinates (x_0,y_0,z_0,t_0) and (x, y, z, t) refer respectively to a system *at rest* and a system with uniform velocity v in the direction of the z-axis and where $\beta=v/c$, c being the velocity of light. Let a particle be at rest in the system (x_0,y_0,z_0,t_0). If waves are to be associated with such a particle it is natural to assume that they are stationary in this system, so that the time shall enter into their mathematical expression only in a factor of the form $\cos 2\pi\nu_0 t_0$, provided that the time origin has been suitably chosen. The constant ν_0 is called the characteristic frequency of the particle. Let us now suppose that the same particle is observed from the system (x,y,z,t). The form of the associated waves in this system may then be determined by means of the Lorentz transformation. Then the factor $\cos 2\pi\nu_0 t_0$ takes the new form $\cos 2\pi\nu(t-z/v)$, where

$$\nu=\nu_0/(1-\beta^2)^{1/2},\quad V=c/\beta=c^2/v.$$

Therefore, to an observer on the system (x,y,z,t), the waves seem to have a frequency ν and to proceed with the phase velocity V. This result appears as a simple and direct consequence of the transformation characterizing the restricted theory of relativity.

It turns out that the *group velocity* of the waves, defined in the way indicated by Lord Rayleigh, is simply the ordinary velocity v of the particle with reference to the system (x,y,z,t).

Some additional hypothesis is necessary to bring the conception of associated waves into connection with the quantum theory; and this is made by supposing that the energy W of the particle and its frequency ν are connected by the equation

$$W=h\nu,$$

where h is Planck's constant.

Such, in its simplest aspect, is the starting point of the new wave mechanics of de Broglie. This is presented, together with some of its immediate consequences, in the third chapter (pp. 34–41) of de Broglie's *Wellenmechanik*. The earlier part of the book consists of an introduction (pp. 1–9) and two chapters devoted to some general preliminaries belonging to the classical mechanics both in its Newtonian and in its relativistic form. The introduction contains a highly interesting and (in the main) non-technical presentation of some of the leading ideas of the new mechanics.

From the simple starting point indicated in the third chapter the author proceeds in the fourth (pp. 42–58) to a general introductory investigation of wave propagation in the new mechanics and thence to a study of the differential equation of mechanical waves in the fifth chapter (pp. 58–67). Here one encounters the now classic differential equation of Schrödinger. In the

sixth chapter (pp. 67–75) there is an account of the relations between classical mechanics and wave mechanics. The seventh chapter (pp. 75–87) is devoted to the principle of interference and the diffraction of electronic streams by crystals: it brings to attention what is perhaps the most remarkable experimental confirmation that the new wave mechanics has yet received. Some related matters are treated in the eighth chapter (pp. 87–94). The ninth chapter (pp. 95–105) is devoted to the simpler aspects of those questions of probability which are forced upon one's attention by the new mechanics and particularly by Heisenberg's principle of uncertainty (already brought before the reader in the fourth chapter). The tenth chapter (pp. 105–116) is devoted to the remarkable wave mechanics of light quanta: it is shown in particular that the Einstein theory of light quanta is an almost immediate consequence of the initial hypotheses of wave mechanics. The Heisenberg-Bohr theory is developed in the eleventh chapter (pp. 116–126), and the bearing of Heisenberg's principle of uncertainty on fundamental questions of experimental possibility is treated in the twelfth chapter (pp. 127–136), while the thirteenth (pp. 137–151) is devoted to a further investigation of wave propagation. Up to this point the author has confined his attention exclusively to the motion of a single particle in a given field. The remaining six chapters (pp. 152–219) treat of the wave mechanics of systems of particles, the general method of exposition being much like that employed in the earlier chapters for the simpler cases.

It is obvious that so brief a book can not contain a full treatment of the whole of the new quantum theory; but it probably serves the purpose of an introduction to the theory better than it would if it were more comprehensive. One who reads this book with understanding will be prepared to make effective use of the original memoirs.

De Broglie's conception of wave mechanics was set forth in his dissertation in 1924. In 1925 W. Heisenberg presented his new quantum mechanics in its earliest form. Birtwistle, at the beginning of the opening chapter of his book named above, places this work of Heisenberg in the foreground of attention. He says: "The origin of the new quantum mechanics is found in a memoir by Werner Heisenberg which marks the beginning of a new epoch; it contains a new idea which has led to extraordinary developments in quantum mechanics during the last two years · · · . In the new mechanics, the equations have a form identical with that of the classical theory, but the variables do not obey the commutative law of multiplication; · · · the quantum conditions of the older theory are here replaced by equations for determining $xy-yx$; these equations contain Planck's constant h."

After an introductory chapter on the origin and development of the new quantum mechanics, Birtwistle devotes six chapters to a discussion of various experimental facts and certain laws or principles which have been directly associated with them. He then introduces in the eighth chapter (pp. 63–70) the Heisenberg quantum mechanics of 1925 with its non-commutative multiplication, develops in the next chapter (pp. 71–78) the theory of Dirac, gives in the tenth chapter (pp. 79–84) Heisenberg's theory of the harmonic oscillator and continues for several chapters with related matters. The work of Schrödinger dominates Chapters XVII to XXII (pp. 139–189); its relation to

the work of Heisenberg is brought out with care, especially in Chapter XX where the Heisenberg matrices are determined from the point of view of Schrödinger's theory. The theory of de Broglie is briefly treated in Chapter XXIII (pp. 191–200). At this stage the more important general ideas are before the reader. In the remainder of the book we continue to have experimental results and theoretical developments presented alternately or interpenetrating each other, as may be found convenient in exposition.

From this outline of Birtwistle's book it will be seen that his exposition is guided by considerations far removed from those which dominate de Broglie's. On account of this great difference the two volumes supplement each other. In Birtwistle's more attention is given to experimental results than in de Broglie's; and theoretical considerations are presented by the two authors in markedly different orders. The book by Weyl is very different in conception and treatment from each of the preceding. It resembles Weyl's *Raum, Zeit, Materie* in plan in that only half of it belongs to the author's field (mathematics) while the other half belongs to physics. The external circumstances leading to its preparation were as follows. In the winter semester of 1927–1928 Zürich was left without a theoretical physicist owing to the fact that Debye and Schrödinger were called elsewhere. Weyl sprang into the breach by transforming his announced lectures on group theory into lectures on group theory and quantum mechanics; and the book before us resulted from the systematic development of the material so organized. The exposition carries with it evidence of the two influences involved in its production. The theory of the representation of continuous groups is developed in such a way as to exhibit its close contact with quantum theory. "From this book, if it attains its goal," so speaks the author, "one should be able to learn the fundamentals of group theory and quantum mechanics; in the mathematical sections the physicist is kept chiefly in mind, in the physical sections the mathematician."

The book falls into five chapters: the first treats the basis of the unitary geometry; the second is devoted to quantum theory and contains what is necessary for an understanding of the ideas and methods of this theory; the third chapter develops the elementary parts of the theory of representations of groups; an application of this theory to quantum physics constitutes the fourth chapter. In these four chapters mathematics and physics alternate. In the fifth chapter, on the other hand, there is a combination of both mathematics and physics in that the theory of permutation groups and representations of them is developed and is applied to important problems in atomic physics.

The book is not well suited to the needs of a reader who would make his first excursion either into group theory or into the new quantum physics. But one who has already become acquainted with the fundamental aspects of these two disciplines will find here a stimulating exposition and will obtain fruitful ideas for continued meditation.

R. D. CARMICHAEL

LEBESGUE ON INTEGRATION

Leçons sur l'Intégration et la Recherche des Fonctions Primitives. By H. Lebesgue. Second edition. Paris, Gauthier-Villars, 1928. xv+342 pp.

A second edition of Lebesgue's book on integration after almost twenty-five years! Is it going to be a treatise on Lebesgue integration, following the lead of some recent books on this subject, or will it be in fact a second edition, i.e. a continuation and bringing up to date of the theme which dominated the first edition? The preface already gives enlightenment on this question. We find that Lebesgue has not yielded to the temptation of going into the details of the Lebesgue type of integration and its application, but is more interested in giving a logical continuation of the first edition of the book. He apologizes for omitting so many things which might be of interest, simply because they do not belong. The first edition is almost entirely contained in this second one, even to the extent of misprints, but the book has grown to almost three times its original proportions.

The new edition still forms one logical whole. The underlying problem and theme of the book is still to be found on the first page: "To find the functions $F(x)$ which admit as a derivative a given function $f(x)$." It is this theme with its variations, extensions and elaborations which forms the subject matter of the book. The author, in true composer fashion, begins with a simple statement of the theme, produces the harmony contained in the researches of Cauchy, which gives the wonderfully esthetic result that every continuous function is the derived function of another continuous function, carries it through the extension to Riemannian and related integrations, the introduction of Lebesgue integration, and as an added and extended feature, treating not only the derived functions but the upper and lower derivates. More and more complex become the variations as we approach the contribution made by Denjoy in his totalization, from which Lebesgue extracts in a triumphant fashion the important theorem on the possible distributions of the values of the derivates of a continuous function. Once again the theme starts simply and casually with a new kind of harmonic structure, the Stieltjes Integral puts in its appearance and through a series of complexities, one is led to a grand finale of the notion of derivatives with respect to functions of bounded variation and a totalization process as applied to these. As one applies a critical analysis to this composition as a whole, one cannot help but admire the skill and artfulness with which the entire structure is built, how each item contributes its share to the whole, how each consideration leads on to further results, and plays its own part in the sequel. It is indeed wonderful how the material drawn from the first edition seems in some farsighted way to have been a fitting preliminary to the later developments and takes its place as an integral and inseparable part of the whole.

The first six chapters of the book are to a large extent identical with the corresponding chapters in the first edition, additions being mainly in the footnotes, and to some extent in the text, where the later chapters seemed to require more extended treatment of topics, and where the intervening years have

brought new worthwhile results. These chapters take one through the Cauchy definition of integration, that is, the integration of continuous functions, the Riemann definition, the geometric definition of integration, functions of bounded variation—this topic extended slightly to include a more extended study of the convergence of the variational sums to the total variation—and a study of the derived numbers of continuous functions and questions of the determination of a function by its derivates.*

These chapters have been discussed in the review of the first edition of the book (this Bulletin, vol. 14 (1908), pp. 501–4). We subscribe to the opinions there expressed, except that in the intervening years the material of the first edition has become a natural part of our mathematical thinking and seems simpler than when first propounded. In going over this section we were rather impressed by the variety of results concerning derivatives and derivates obtainable without the use of integration, e.g., the theorems which give the leeway in the determination of a function by its derivates, the theorem that a uniformly convergent sequence of derivative functions is again the derivative of a continuous function, and so on. Although Lebesgue later on makes a point of obtaining results on derivatives by studying the inverse process, he gives in these early chapters a fine illustration of how much can be accomplished by direct methods.

The seventh chapter gives an exposition of the Lebesgue method of integration, and follows the manner of treatment in the first edition, that is, starting with the given properties of integration, he derives a form which these integrals must take. The main additions in this chapter are more explicit proofs of the fact that the integral obtained by using the properties specified actually does have the desired properties, an addition which is highly desirable. Of the many other methods of defining the Lebesgue type of integration, he singles out one of the early definitions of W. H. Young, which follows the Riemannian type of definition, subdivisions being into measurable sets instead of intervals. It is perhaps interesting to remark that Lebesgue's definition lends itself most readily to proofs of properties relating to additivity of the integral as a function of sets, while the Young type lends itself to the additivity as a function of functions.

The eighth chapter, entitled *The indefinite integral of summable functions*, is devoted to an exposition of completely additive functions of measurable sets, suggested by the fact that the integral is such a function of sets. The author distinguishes between three types of indefinite integral, $\int_a^x f(x)dx$+const.; $\int_\alpha^\beta f(x)dx$, (α, β) being any subinterval of (a, b); and $\int_E f(x)dx$ where E is any measurable subset of (a, b). He amuses himself by playing with the words indefinite and definite, e.g., "It is clear that in the expressions definite and

* It seems unfortunate that some of the misprints and errors of the first edition were not corrected in the second. The reviewer is puzzled as to why (p. 78) the function $x^2 \sin(1/x)$ should be cited as an example of a function of bounded variation with unbounded derivative, why (p. 102) the continuous function $x(x^2 \sin(1/x))'$ is not integrable D, i.e. the derivative of a continuous function, and why (p. 139) $m_{i,4}(\phi \geqq \alpha)$ is continuous on the left in α since it is not in general true that if the sets E_n form a monontonic increasing sequence converging to E, then the lower measure of E_n approaches that of E.

indefinite integral, indefinite is not in the sense of infinite, but of not defined. Definite has the sense determined. $\int_\alpha^\beta f(x)dx$ is definite if the interval of integration is defined, but indefinite when (α, β) is undefined, not definite, not determined, unknown, variable." And later in a footnote (p. 195), in which he takes exception to Denjoy's introduction of new words like "épaisseur" as opposed to "density at a point,"* he concludes with "however that may be the reader who has followed me up to this point will not be troubled by the question of words, since he has consented to engage in investigating the definition of the definite integral, which might seem ridiculous, and the definition of the indefinite integral, which remains indefinite even after it has been defined, which is a little humiliating."

The main portion of this chapter is devoted to the separation of an absolutely additive function of sets into three parts, the function of the discontinuities, the absolutely or completely continuous portion, and the function of singularities, together with the properties of the sets of singularities, the treatment suggested to a large extent by the work of de la Vallée Poussin. What worries one particularly in this chapter is that it is not stated explicitly at the beginning of the investigation to what class of sets these additive functions apply, and it is only by reading between the lines that one concludes that the author is interested in additive functions of measurable sets.

Chapter IX is concerned mainly with the derivative properties of the indefinite Lebesgue integral. By setting up the integration process for a derivate of a continuous function he obtains the result that a continuous function of bounded variation has a finite derivative except at a set of measure zero, and derives the relation between the variation of the function on an interval and the integral of its derivative, yielding the well known necessary and sufficient conditions that a function be an indefinite Lebesgue integral as well as the relationship between a function and its integrable derivatives. The method of procedure, while perhaps consistent and logical in its setting is rather involved, playing tag with ϵ's approaching zero, and M's approaching infinity, to an extent to make it seem almost uncertain that the theorems have actually been proved. From the point of view of elegance the method of de la Vallée Poussin, involving the majoring and minoring functions, is far superior. The later parts of this chapter extend the theorem on the existence of a derivative of a function of bounded variation to the discontinuous case, the method being via results on integrals. There then follows a brief exposition of derivatives at a point as defined by the quotient of a set function by the measure of a set. There is a delightful explanation of how the desire to have

* When it comes to a question of terminology, the reviewer has his doubts as to the advisability of introducing a new terminology for each new method of defining integration. The use of the word summable, transliterating the French "sommable," for Lebesgue integration, interferes with the word usually applied to a process relating to infinite series. Is Lebesgue integration a summability process applied to functions not Riemann integrable? And then the word totalization for Denjoy's definition. Why total? I prefer to use the word integration for all such processes and, if distinctions are needed, to characterize them by special names, Riemann, Lebesgue, Denjoy, Stieltjes.

the values of the function near a point be the main contributors in the value of the derivative leads inevitably to the notion of order of a set at a point. The chapter closes with a brief treatment of length of arc, a subject which assumes a place in the background in this edition.

The tenth chapter is devoted to a presentation of the subject of totalization, i.e., the Denjoy extension of the Lebesgue integration. The presentation of the material is such that one gains an insight into this process which the writings of Denjoy and others have not succeeded in conveying. Perhaps the easy steps through which the ultimate exposition of the totalization process in all its generality is reached contribute no small share. In order to get the reader familiar to some extent with the transfinite operational process involved, we have as the first section of this chapter a proof of the Baire Theorem concerning the nature of functions of the first class. This theorem illustrates two points which are made fundamental to the later considerations, (a) that a well ordered set of closed sets each of which is non dense on the preceding terminates after a denumerable number of steps, (b) if for a given operative process O, it is known that for every closed set E belonging to (a, b) there exists a subinterval (l, m) containing points of E on which the operation O may be defined, then the operation O can be defined on (a, b).

In discussing totalization the first type of function considered is assumed to be the finite valued derivative $f(x)$ of a continuous function $F(x)$. For such a function $f(x)$ it is proved that for every closed set E there exists an interval (l, m) containing points of E, so that $f(x)$ is bounded on the part of E in (l, m) and the sum $\sum [F(\beta) - F(\alpha)]$ extended over the intervals (α, β) complementary to E relative to (l, m) is absolutely convergent, making it possible to define F on the interval (l, m) if it is known on the intervals complementary to E, by the formula $\int_E f(x)dx + \sum [F(\beta) - F(\alpha)]$. Then come finite valued functions f, which are known to be one of the derivates (right hand, upper or lower, or left hand, upper or lower) of a continuous function, and in that case the boundedness of f on the part of E in (l, m) is replaced by the summability. The process of totalization then divides itself into two parts, one part to enable one to extend it step by step over a set of intervals each containing the preceding, the other involved in the formula quoted above. There follow theorems giving conditions under which a function is totalizable, and also some giving necessary and sufficient conditions in order that a continuous function $F(x)$ be an indefinite total, the most elegant of these latter theorems being that for every closed set E the function $G(x)$ equal to $F(x)$ on E and linear on the complement of E is absolutely continuous in some interval containing points of E. It is to be noted that the totalization process which is treated here is not the one originally proposed by Denjoy. This is a weaker definition and therefore more generally operative. The first totalization process demanded certain absolute convergences, including the convergence of the oscillations (not variations) of the function on the complementary intervals of closed sets. This first type of totalization has properties much more closely allied with those of the simpler integration. In order to get derivative properties for the general totalization it is necessary to introduce the notion of approximative derivative at a point, i.e. the derivative of a function $F(x)$ on a set of density 1 at the point. The final section of the chapter gives the elegant theorem of Denjoy on the

possible distribution of the values of the four derivates of a continuous function, together with the interesting result that if one knows that a finite valued function $f(x)$ is at each point one of the four derivates of a function $F(x)$, not always the same derivate, then the function $F(x)$ is determined from $f(x)$ by the totalization process. Totalization, then, as the indirect process can accomplish for the derivatives of a continuous function results derivable also by studying these derivates directly. Obviously it does not extend to discontinuous functions, where direct methods are more powerful and effective. Totalization does accomplish about all that one has a right to expect for the inversion of finite valued derivates, the path for the inversion of the infinite valued derivate being obviously blocked by the inability to distinguish between infinites.

The chapter on Stieltjes integrals begins with a discussion of what might be called the Stieltjes integral, where, in $\int f d\alpha$, f is continuous and α of bounded variation. Thus $\int f dx$, f continuous, might be called the Cauchy integral. In listing the properties of these integrals Lebesgue strangely enough omits explicit mention of the linearity in α which is used perhaps more than any of the others Much time and effort is devoted to reducing the Stieltjes integral to a Lebesgue integral by transformation, but Lebesgue fails to comment on the fact that his own definition of integral is really a Stieltjes integral, $\int_m^M y d\mu(y)$ where $\mu(y)$ is the measure of the set $E(f>y)$, and (m, M) is an interval enclosing the interval of variation of f. It seems to me however that in such transformations one loses sight of important properties. It certainly is not a very easy matter to deduce properties of Lebesgue integration from its expression as a Stieltjes integral, and the properties of the function $\mu(y)$ associated with a given function. The marvel is how many of the properties and extensions of Stieltjes' integrals can be derived from their Lebesgue integral transform.

There is a lack of clearness in the treatment of the so-called Riemann-Stieltjes integral; why a new terminology is necessary is not obvious. To the reviewer the proof of the theorem that a necessary and sufficient condition that a function f be Stieltjes integrable with respect to a function of bounded variation α is that the set of points of discontinuity of f have a total variation zero with respect to α, is not at all convincing. There seems to me to be a difficulty regarding the limits processes involved. Essentially there are, in integration definitions of the Riemann type, two methods of approach, one as to a norm, viz. the maximum of the length of the intervals in a subdivision, the limit being in the sense that the norm approaches zero, the other a limit as to subdivisions, monotoneity of approach being obtained by assuming that each subdivision is a redivision of the preceding.* The Riemann integration is an instance of the first type, the Darboux integrals exist after the second manner, and the conditions of integrability via the upper and lower integrals are dependent upon the proof of the fact that the Darboux integrals are also existent as limits as to a norm. The total variation of a function exists as a limit after the second manner, and after the first only if the function has only regular points, that is, for every x, $\alpha(x)$ lies between $\alpha(x+0)$ and $\alpha(x-0)$. The upper and lower in

* See for instance, E. H. Moore, Proceedings of the National Academy, vol. 1 (1915), pp. 628–32, American Journal of Mathematics, vol. 44 (1922), pp. 102–114.

tegrals in a Stieltjes integral exist after the second manner, but in this case there is in general no theorem corresponding to the Darboux theorem unless α is continuous. Lebesgue modifies the norm definition by requiring that the points of discontinuity of α be gradually made to be the points of division, so as to secure an equivalent of the second mode of definition, which method of approach is obviously limited to the case in which the points of discontinuity of the function α are denumerable.

After a brief reference to the notion of Stieltjes integrals in the applications, there follows the definition of the derivative of a function with respect to a function of bounded variation, boldly set down as the limit of $\Delta F/\Delta\alpha$ as x approaches x_0, without any particular worry about the behavior of the function α except for its bounded variation character. This leads up to the notion of a totalization process extended to a Stieltjes integral. Lebesgue seems loath to abandon the condition that the function α be of bounded variation, because "one cannot attach to every continuous function an integral with respect to α if α is not of bounded variation." But just why continuous functions should be singled out in this way is not clear. This section is to some extent rather sketchy but on the other hand very suggestive, and will undoubtedly give rise to interesting further developments. It forms a fitting close to the main part of the book.

The book closes with an appendix of about twenty-five closely printed pages, on the subject of transfinite numbers. It gives an interesting introduction to this topic, interspersed with comments and cogitations on the subject and its applicability. There are many worthwhile things. Perhaps the nature of some of the cogitations will be evident from some quotations. In connection with the reasoning by transfinite induction he says: "Must one then accept this method of demonstration? Not at all, it is up to each one to decide for himself whether or not he is satisfied with reasoning by transfinite induction. Further as almost everything in mathematics has been written only for those who admit the reasoning by ordinary induction so certain passages of this book are written only for those who permit reasoning by transfinite induction." In discussing the matter of replacing proofs by the use of transfinite induction by proofs avoiding this method, he distinguishes sharply between the *theorem* of Cantor-Bendixson, that every closed set is the sum of a perfect set and a denumerable set, and the *problem* of Cantor-Bendixson to decompose a closed set into the sum of a perfect set and a denumerable set; similarly the *theorem* of Baire, that every function of the first class is pointwise discontinuous on every perfect set, and the *problem* of Baire, for a function pointwise discontinuous on every perfect set, to set up a sequence of continuous functions of which it is the limit. He continues: "In the researches of Denjoy, the very definition of the operation of totalization requires the transfinite, and one cannot hope to avoid its use. But nothing prevents one from hoping that it is possible to replace the operative process of Cantor-Bendixson, of Baire, and of Denjoy by non-transfinite processes permitting one to solve the problems of Cantor-Bendixson, Baire, and 'primitive' functions respectively. In fact it is possible to demonstrate the theorem of Cantor-Bendixson, and of Baire, and to solve the problem of Cantor-Bendixson, without the use of the transfinite induction."

T. H. HILDEBRANDT

SHORTER NOTICES

Einführung in die Algebra. By Otto Haupt. Vols. I–II. Leipzig, Akademische Verlagsgesellschaft, 1929. 663 pp.

The domain of algebra has in recent years been enriched with a great number of German textbooks of high quality. I need only mention the books by Perron, Fricke, Hasse, and Bieberbach's edition of Bauer's algebra. The new algebra by Haupt might therefore at first glance seem somewhat superfluous. A closer study, however, reveals that it as an unusually successful attempt to present the foundations of modern abstract (commutative) algebra within the scope of a textbook.

The book is mainly based on the fundamental ideas of Steinitz set forth in his paper *Algebraische Theorie der Körper* (Journal für Mathematik, vol. 137 (1910), pp. 167–309). It contains numerous references to newer literature and several investigations are here for the first time presented in a textbook. The lucidity of its style makes it a convenient and attractive introduction to some very important parts of modern algebra.

The book starts with a discussion of the axioms of algebra, and then rings, domains of integrity and fields are introduced and the existence of quotient-fields is proved for all domains of integrity. In the third chapter the foundations of group theory are laid down; then follows an investigation of the axioms of divisibility, Euclid's algorithm, and unique decomposition in prime factors. Furthermore the classes of remainders are analysed and the abstract fields are classified according to their *characteristic* as done by Steinitz.

The next chapters in volume I are not very different from the representation of these matters in ordinary textbooks. The only difference is that the coefficients are always elements of an arbitrary abstract field, and that the fields of characteristic $p>0$ in certain cases may cause exceptions. The headings are: Transcendental adjunctions (that is, fields of rational functions of one or more variables), symmetric functions, linear equations, divisibility of polynomials in fields and domains of integrity. An interesting paragraph deals with the determination of the prime-function decomposition in a finite number of operations. The method applied is a direct generalization of Weber's method for algebraic fields.

The existence of abstract fields in which a given polynomial is equal to a product of linear factors is then proved; I remark, en passant, that the proof of this theorem is considerably simpler than the proof for the ordinary fundamental theorem of algebra, which states that *every* polynomial with coefficients in the complex field is equal to a product of linear factors in the same field.

All prime-functions can be divided into two classes: An irreducible polynomial belongs to the first class, when all its roots are different; if not, it belongs to the second class. A field in which all polynomials are of the first class is called *perfect.* The complete Galois theory can be extended to these fields. A field can only then be imperfect, when it has a characteristic $p>0$ and there exists an element a such that $\sqrt[p]{a}$ is not contained in the field.

The last chapters of volume I deal with equations of third and fourth degree and other special equations. In volume II the ordinary fundamental theorem of algebra is proved. Then the Galois theory for all polynomials of the first class is very extensively developed and additional considerations solve the difficulties for polynomials of the second class. This volume contains various subjects new to textbooks. I will only mention the mutual reduction of polynomials and the determination of equations without *affect*. In an addendum Krull studies the Galois theory of algebraic fields of infinite rank, generalized Abelian groups, and their application to the theory of matrices and elementary divisors.

The author assumes that the book should be used for a first textbook in advanced algebra. It seems to me, however, that such a book can only be appreciated by students who already have a rather thorough knowledge of algebra; for a beginner in the subject many of the considerations might look like unnecessary complications and formalities. In general, an axiomatic theory very often has a depressing influence on a reader who is not familiar with the general plan of the building he is going to cooperate on. By taking this attitude the author could have omitted some of the more elementary parts of the book and saved space for other items like, for example, divisibility and the theory of rings. This is, however, no serious reproach to this remarkable book.

ØYSTEIN ORE

Mathematical and Physical Papers. By Sir Joseph Larmor. 2 volumes. Cambridge, University Press, 1929. Vol. 1, xii+679 pp., vol. 2, xxxii+831 pp.

The purely mathematical papers form but a small part of the author's work; they were written mostly in the eighties and deal with the geometry of quadric surfaces, transformations of integrals, and the formal aspects of various methods in mathematical physics.

Foremost among the author's contributions to physics are the three, now classical, papers on *A dynamical theory of the electric and luminiferous medium*. Adopting for the electric energy function the form first used by MacCullagh in optics, the author develops the most complete ether theory of electromagnetism ever devised, and correlates it with thermodynamics, kinetic gas theory and other branches of theoretical physics. Much space is given to the theory of the electron, and there are many results in common with the investigations pursued by Lorentz about the same time by different methods. One of the best known of Larmor's results, which has recently acquired added significance in quantum mechanics, is the "Larmor precession" of the orbit of an electron in a constant magnetic field.

Next in importance to the electromagnetic papers are those on general thermodynamics and the theory of gases. There are also a number of papers on geophysical questions, and various reports and addresses.

Larmor handles his mathematics with elegance, but the outstanding features of his work are the clarity and power of his physical reasoning, and the definiteness and comparative simplicity of his mechanical models. A valuable feature of the present edition is the extensive series of historical and critical

notes contributed by the author. The printing is in the familiar style of the Cambridge University Press, with the added luxury of a marginal index.

T. H. GRONWALL

Euclides Danicus. By Georg Mohr. Amsterdam, 1672. Copenhagen, Høst, 1928. Price $2.50.

Towards the end of the eighteenth century Mascheroni arrived at the amazing conclusion that all euclidean constructions may be executed with compasses alone, without the aid of the ruler. This was the starting point for the study of the rôle of instruments in geometric constructions, so successfully carried out during the nineteenth century. But Mascheroni's book *Geometria del Compasso,* (Pavia, 1797), remained the standard work on Mascheronian constructions. It has been superseded only very recently (A. Quemper de Lanascol, *Géométrie du Compas,* Paris, Blanchard, 1925).

Had Mascheroni died in childhood would science have been deprived forever of Mascheronian geometry? Usually such a question is a moot one. Not so in this case. In 1672 a Danish mathematician, Georg Mohr, published a book in Dutch and in Danish simultaneously, which contains Mascheroni's basic result and a good many of his problems. Little is known about Mohr. Leibnitz mentions him in one of his letters. Mohr's book passed completely unnoticed. Mascheroni says explicitly in the preface of his *Geometria* that he knows of no previous work of this kind. There is no reason to doubt his word.

Due to the efforts of J. Hjelmslev, the Danish Scientific Society has published a facsimile copy of the Danish edition of Mohr's book together with a German translation of it. The Danish Society deserves to be congratulated for having rescued this book from oblivion. While the book does not add to our geometric knowledge, it is an interesting historical document, in more than one respect. The typography of the book is excellent.

N. A. COURT

Hoehere Mathematik. Teil I. *Differentialrechnung und Grundformeln der Integralrechnung, nebst Anwendungen.* By Rudolf Rothe. 3d edition. Leipzig, B. G. Teubner, 1930. vii+189 pp.

This edition differs but little from the first and second editions (reviewed in this Bulletin, vol. 31, pp. 556–7, vol. 33, p. 791). The changes consist of a few alterations and corrections in the text, the addition of a paragraph relating to moments, which might well have received more extended treatment in volume II, a paragraph in the discussion of Taylor's formula, and a paragraph deriving Lagrange's rule for maxima and minima of functions of more than one variable with supplementary condition, in which the author assures us that the vanishing of the jacobians: $\partial(f,g)/\partial(x,z)$ and $\partial(f,g)/\partial(y,z)$ is equivalent to the existence of a *constant* λ such that $\partial f/\partial x+\lambda\partial g/\partial x=\partial f/\partial y+\lambda\partial g/\partial y=\partial f/\partial z+\lambda\partial g/\partial z=0$.

T. H. HILDEBRANDT

Analytische Geometrie. By L. Bieberbach. Leipzig and Berlin, Teubner, 1930. iv+120 pp.

This little book (volume 29 of Teubner's Mathematische Leitfäden) presupposes an elementary knowledge of the subject. The novelty of the treat-

ment lies in the introduction from the outset of vectors and matrices. The treatment of the geometry of the triangle by vectors is very elegant, as is the application of matrices to the study of orthogonal coordinate transformations and the classification of quadrics.

T. H. GRONWALL

Methoden der Praktischen Analysis. By Fr. A. Willers. Göschens Lehrbücherei, Band 12. Berlin and Leipzig, de Gruyter, 1928. 344 pp.

This book treats numerical, graphical, and some instrumental methods of practical analysis. While the most important is the numerical, by which an approximation to any desired degree of accuracy may be reached, the author has felt the importance of treating the less precise methods because they frequently give results which are sufficiently accurate for the purpose in hand and in other cases provide easily a first approximation to more refined results by the numerical method. Lack of space has made it necessary to limit the applications in neighboring fields and the author regrets particularly that he has had to omit almost everything from mathematical statistics.

The first chapter considers the general problems involved in calculations with approximate numbers and describes the mechanical aids to computation, particular attention being given to various special types of coordinate paper. The second chapter, which is fundamental to those which follow, develops standard methods of interpolation including numerical differentiation and integration. In the third chapter we find a treatment of approximate integration and differentiation by various formulas and types of integrating machines. The fourth chapter is devoted to methods of approximating the roots of equations, including systems of linear equations and the complex roots of algebraic equations. The last paragraph gives a brief treatment of linear difference equations. Empirical formulas and curves, particularly those involving periodic functions, are described in Chapter Five. The sixth chapter explains graphical and numerical methods for obtaining approximate solutions of ordinary differential equations.

The last four chapters are independent and each is based on Chapter Two alone. This volume is to be recommended as a convenient reference book for any one confronted by a problem the solution of which requires the approximate methods listed above.

W. R. LONGLEY

Mathematische Existenz. By Oskar Becker. Untersuchungen zur Logik und Ontologie mathematischer Phänomene. Halle, Niemeyer, 1927. 370 pp.

This appears as a "Sonderdruck aus: Jahrbuch für Philosophie und phänomenologische Forschung, Band VIII," and as such has a double pagination. The work is a critical, searching study of the problem of mathematical existence in its present controversial status. It should be read in connection with the recent prior articles of Hilbert, Brouwer, Weyl and others. One is plunged in the first few pages into the present conflict concerning the foundations of mathematics as championed by Brouwer for Intuitionalism, and by Hilbert for Formalism. It is of significance that there is no present

dispute between these contenders in regard to the subject of finite mathematics as distinguished from what Hilbert calls "metamathematics."

"The first point to be mentioned in this connection is the fundamental thesis of the necessity of a common establishment of logic, arithmetic (with combinatorial analysis) and the theory of aggregates, and indeed with limitation to the finite. This thesis is directed against the attempt of the 'Logistikers' (Peano, Frege, Russell, etc.) to derive arithmetic and the theory of aggregates from formal logic. That this is impossible and indeed worse than impossible since more closely viewed it is meaningless, such is the common conviction of Brouwer, Weyl and on the other side Hilbert." Thus concerning any countable sequence whose successive terms are given by an explicit rule of construction, no dispute here arises. The only features of disagreement occur for uncounted sets, or incompletely defined sequences. Brouwer insists upon the essential time element in any concept of unending sequence where no constructive rule is present. Hilbert admits that metamathematics can only be established by the use of a new order of ideas. He does not depend upon any principle of justification which requires explicitly time or human mental processes. As a consequence metamathematics can be thought of as studied purely formally, without regard to content, it being at least unnecessary in view of the other fruitful methods of approach, to follow so far the weary way of objective recognition of content.

For Brouwer, one cannot assert with regard to an unended sequence that the sequence (thought of as completed) either does or does not contain an element which has not yet been observed to occur in the sequence. The method of excluded middle, and in general reductio ad absurdum as a method of proof is not logically available in such cases. For Hilbert by means of an axiomatic approach the existence of an explicit construction reduces the subject to the realm of the finite or mathematics proper, but the logical existence of objects may be inferred whose construction remains undetermined. In particular Hilbert provides what is claimed as a proof that no contradiction can arise from metamathematical proofs of a certain sort. Hilbert 's proof itself makes free use of reductio ad absurdum, and so is unacceptable to the intuitionalists, even if they were to permit the use of the title "object" to what is known, as to its existence and properties, only by axioms.

The author may be regarded as using this conflict more or less as a text. The views of both sides are elaborated with clearness and sympathy but many trenchant comments (not all original of course) are directed at vulnerable points. Then the whole topic is studied historically, with special reference to Plato, Aristotle, Descartes, Leibniz, Kant, Cantor, Russell. No brief summary would be just to the writer.

Approximately half of the work (the second half) is devoted to the philosophical problems of valuation in terms of human activity as distinguished from ontology. This part may prove of minor interest to many mathematicians who can study the first half with profit and enjoyment.

The author settles no major issues, and champions no special cause, but he here provides an effective and discriminating presentation of some logical mathematical problems of present interest. If the style seems heavy, at least there is no hiding behind sonorous terms. Only the "logistikers" seem hope-

lessly outmoded. No special sympathy is offered to those who would justify arithmetic on a contentless axiomatic basis. The tone of the discussion is not adjusted to the level of anyone so naïve as to suppose that therefore the theorems of arithmetic are not assumed as logically derivable from a suitable system of postulates such as that framed by Hilbert himself.

A. A. BENNETT

Lezioni di Geometria Analitica e Proiettiva. By Annibale Comessatti. Padova, Casa editrice A. Milani, 1930. xv+462 pp.

The present volume, most of which had been used repeatedly by the author in mimeograph form previously, represents the work done in algebraic and projective geometry by the Italian university students during the first and second years. Probably no other country adheres so closely to the old Greek methods, nor guards so jealously the rich and varied traditions of its past. And in the present case, without forfeiting anything of this heritage, the book is decidedly down to date, not only in content, but also in method.

The subject is introduced as in our own elementary texts in plane analytic geometry, with numerous figures, and many examples left for the student, but presented at a much more rapid pace. Determinants are used freely from the beginning, and projective (affine) coordinate systems precede metrical ones. Properties of lines, planes, conics and quadric surfaces are derived in the first chapter of about 60 pages. This is followed by one on algebraic vector analysis in two and three dimensions, with a skillful blending of the method and results of the first chapter with those of the second.

Chapter three introduces projective geometry in the usual way, with the immediate extension to infinite points, lines and planes. After establishing the foundations synthetically, coordinates are employed: point coordinates, plane coordinates, line coordinates (Plücker), including complexes, congruences and ruled surfaces. Then follows a generous appendix on complex geometry.

A large chapter on curves and surfaces introduces the calculus and employs it to get metrical properties of plane algebraic and transcendental curves, of surfaces and of space curves, including an extensive treatment of quadric surfaces, reguli, asymptotic cones, etc.

The work is written in the peculiar limpid style of Comessatti, and is printed on thin opaque paper, making a convenient volume. One objection is that the exercises are printed in small type—pages of them at a time—and contain a wealth of valuable material that one can not afford to overlook.

This book would not be a suitable text for beginners in either analytic or projective geometry, according to the American methods of instruction, but is particularly valuable for reference and comparison after one has reached the maturity necessary to appreciate it. It is an eloquent commentary on the intense instruction given to the precocious Italian youth.

VIRGIL SNYDER

NOTES

The Oxford University Press has arranged to continue the Quarterly Journal of Mathematics and The Messenger of Mathematics as a single journal, to be called The Quarterly Journal of Mathematics (Oxford Series). The editors will be T. W. Chaundy, E. G. C. Poole, and W. L. Ferrar.

The former students of Professor E. L. Lindelöf, of the University of Helsingfors, have published a volume in his honor entitled *Commentationes in Honorem E. L. Lindelöf a Discipulis Editae.*

The Mathematical Society of Kharkow passed its fiftieth anniversary in September, 1929. A meeting was held to celebrate this anniversary on June 26, 1930, during the session of the Congress of Mathematicians of the USSR held at Kharkow. Foreign mathematical societies were invited to send delegates.

The Sociedad Cientifica "Antonio Alzate" will hold a scientific congress in the City of Mexico, September 9–14, 1930. There will be a section devoted to the mathematical sciences. Foreign mathematicians are invited to attend.

The Reale Accademia Petrarca di Lettere, Scienze ed Arti has presented a gold medal to Professor Francesco Severi.

The class of physical sciences of the Royal Academy of Bologna announces the following subjects for its Adolfo Merlani prize: *The iteration of rational functions*; *Conformed representation.* Competition closes December 31, 1930.

Cambridge University has awarded Smith's prizes to R. E. A. C. Paley and J. A. Todd, and Rayleigh prizes to W. R. Andress and L. C. Young, all of Trinity College.

The Duddell medal of the London Physical Society for 1930 has been awarded to Professor A. A. Michelson, of the University of Chicago. This medal is given for the invention or design of scientific instruments or the discovery of material used in their construction.

The Royal Society of Edinburgh has awarded its James Scott prize for the period 1927–30 to Professor Niels Bohr.

Dr. A. T. Doodson, of the Liverpool Observatory and Tidal Institute, has been awarded the Gray Navigation Prize of the Society of Arts for his work on the analysis and prediction of tidal currents.

The Daniel Guggenheim gold medal for aeronautics has been awarded to Professor Ludwig Prandtl, of the University of Göttingen, for "pioneer and creative work in the theory of aerodynamics."

The New York Academy of Sciences announces the following subject for its A. Cressy Morrison Prize I: *What may be proved from our present knowledge*

as to the possibility or impossibility of released intra-atomic energy constituting an important source of solar and stellar energy. Competing memoirs should reach the Secretary of the Academy by November 1, 1930.

Professor A. E. H. Love, of Oxford University, has been elected correspondent of the Paris Academy of Sciences in the section of mechanics, as successor to the late Sir George Greenhill.

Professor A. S. Eddington has been elected president of the London Physical Society.

Professor T. P. Nunn has been raised to the knighthood.

Professor Albert Einstein has been elected a foreign member of the American Philosophical Society.

The following have been elected members of the National Academy of Sciences: Professor J. W. Alexander, of the department of mathematics of Princeton University; Professor Harry Bateman, of the department of mathematics of the California Institute of Technology; Professor P. S. Epstein, of the department of mathematical physics of the California Institute of Technology.

Professor Enrico Bompiani, of the University of Rome, has recently delivered the following lectures as visiting lecturer of the American Mathematical Society: Differential Equations and Projective Geometry, at Harvard University on May 14; Differential Equations and Projective Geometry, at Columbia University on May 16; A Contribution to the Geometry of Paths, at Princeton University on May 19; Projective Differential Geometry and the Italian School (two lectures), at Cornell University on May 21, 22; What is Geometry, at the Ohio State University on May 28; Italian Contributions to Modern Mathematics, at the University of Iowa on June 30; What is Geometry, Italian Contributions to Modern Mathematics, at the University of Chicago on July 1, 8. During the present Summer Quarter, Professor Bompiani is lecturing at the University of Chicago on Analytic Projective Geometry and on Projective Differential Geometry of Hyperspaces. From October 5 to November 12, he will lecture at Johns Hopkins University on the Differential Equations of Projective Geometry.

Professor Oswald Veblen, of Princeton University, delivered the annual William Lowell Putnam Lecture at Harvard University on April 10 on the subject *Five-dimensional relativity.*

Professors P. W. Bridgman, of Harvard University, and Stephen Timoshenko, of the University of Michigan, are among the members of the newly formed scientific advisory board of the Westinghouse Research Laboratories.

Dr. J. Frenkel, professor of theoretical physics at the Polytechnic Institute of Leningrad, will lecture on wave mechanics and conduct a seminar on problems in modern physics at the University of Minnesota during the academic year 1930–31.

Dr. Karl Menger, of the University of Vienna, has been appointed lecturer in mathematics at Harvard University for the first half of the year 1930–31.

Dr. J. A. Schouten will deliver a series of informal lectures at Harvard University in the fall of 1930.

The following mathematicians have been awarded Guggenheim fellowships for 1930: Professor A. N. Gándara, of the National Preparatory School of Mexico, for studies in differential geometry and harmonic analysis at the Massachusetts Institute of Technology; Professor H. S. Vandiver, of the University of Texas, for research in Europe on Fermat's Last Theorem, laws of reciprocity, and related topics in the theory of algebraic numbers.

Professor Theodor Vahlen, of the University of Greifswald, has been appointed professor of mathematics at the Vienna Technical School.

Professor O. Volk, of the Lithuanian University in Kovno, has been appointed associate professor of mathematics at the University of Würzburg.

Mr. A. E. Ingham, reader in mathematics at the University of Leeds, has been elected a fellow and lecturer at King's College, Cambridge.

Professor C. L. Arnold, of the Ohio State University, has retired from active teaching. He was given the rank of professor emeritus.

Dr. L. A. Bauer, director of the Department of Terrestrial Magnetism at Washington, has retired.

Dr. J. H. Bushey has been appointed to an assistant professorship in mathematics at Hunter College of the City of New York.

Assistant Professor W. F. Cheney, of Tufts College, has been appointed head of the department of mathematics at the Connecticut Agricultural College.

Dr. Alonzo Church has been appointed assistant professor of mathematics at Princeton.

Dr. T. F. Cope has been appointed assistant professor and head of the department of mathematics at Marietta College.

Associate Professor Louise D. Cummings, of Vassar College, has been promoted to a professorship of mathematics.

Dr. H. B. Curry has been appointed assistant professor of mathematics at Pennsylvania State College.

Dr. Marguerite D. Darkow has been promoted to an assistant professorship in mathematics at Hunter College of the City of New York.

Dr. J. Douglas has been appointed assistant professor of mathematics at the Massachusetts Institute of Technology. In April, 1930, Dr. Douglas presented his work in the geometry of paths before Professor Bompiani's class at the University of Rome.

Professor W. E. Edington, of Purdue University, has been appointed professor and head of the department of mathematics at DePauw University.

Dr. B. P. Gill has been promoted to an assistant professorship of mathematics at the College of the City of New York.

Dr. E. L. Hill has been appointed assistant professor of theoretical physics at the University of Minnesota.

Assistant Professor B. P. Hoover, of the Carnegie Institute of Technology, has been promoted to an associate professorship of mathematics.

Dr. Jewell C. Hughes has been appointed to an assistant professorship in mathematics at Hunter College of the City of New York.

Assistant Professor R. P. Johnson, of the Carnegie Institute of Technology, has been promoted to an associate professorship of mathematics.

Assistant Professor R. B. Lindsay, of Yale University, has been appointed associate professor of theoretical physics at Brown University.

Associate Professor P. H. Linehan, of the College of the City of New York, has been promoted to a professorship of mathematics.

Associate Professor M. L. MacQueen, of Southwestern University, has been promoted to a professorship of mathematics.

Assistant Professor Florence M. Mears, of Pennsylvania State College, has been appointed assistant professor of mathematics at George Washington University.

Dr. T. W. Moore has been appointed assistant professor of mathematics at Indiana University.

Associate Professor Marston Morse has been promoted to a professorship of mathematics at Harvard University.

Associate Professor E. J. Oglesby, of Washington Square College, New York University, has been promoted to a professorship of mathematics.

Dr. G. A. Parkinson has been promoted to an associate professorship of mathematics in the extension division of the University of Wisconsin at Milwaukee.

Dr. I. I. Rabi has been promoted to an assistant professorship of theoretical physics at Columbia University.

Mr. L. H. Rice, of the Massachusetts Institute of Technology, has been promoted to an assistant professorship of mathematics.

Dr. J. H. Roberts has been appointed adjunct professor of mathematics at the University of Texas.

Assistant Professor George Rutledge, of the Massachusetts Institute of Technology, has been promoted to an associate professorship of mathematics.

Mr. George Sauté has been appointed assistant professor of mathematics at Cleveland College.

Mr. C. N. Shuster has been promoted to an assistant professorship of mathematics at the State Teachers College at Trenton, N. J.

Dr. C. H. Smiley has been appointed assistant professor of mathematics at Brown University.

Mr. I. S. Sokolnikoff has been promoted to an assistant professorship of mathematics at the University of Wisconsin.

Dr. E. H. Taylor has been promoted to a professorship of mathematics at the Eastern Illinois State Normal College.

Assistant Professor R. W. Veatch, of Ursinus College, has been appointed assistant professor of mathematics at the University of Oklahoma.

Assistant Professor J. L. Walsh has been promoted to an associate professorship of mathematics at Harvard.

Dr. Dorothy W. Weeks has been appointed professor of physics at Wilson College.

Assistant Professor E. A. Whitman, of the Carnegie Institute of Technology, has been promoted to an associate professorship of mathematics.

Associate Professor D. V. Widder, of Bryn Mawr College, has been promoted to a professorship of mathematics.

Associate Professor Mabel M. Young, of Wellesley College, has been promoted to a professorship of mathematics.

The following appointments to instructorships are announced:
Catholic University of America: Mr. E. J. Finan;
Columbia University: Mr. Samuel Borofsky, Dr. A. B. Brown;
Cornell University: Dr. L. A. Dye;
College of the City of Detroit: Mr. W. L. Duren, Jr., Mr. H. H. Pixley;
Harvard University: Dr. J. J. Gergen and the following part-time instructors: D. P. Adams, A. E. Anderson, J. S. Frame, C. B. Morrey, L. T. Moston, S. B. Meyers, G. B. Price, G. B. VanSchaak, Hassler Whitney;
Hunter College: Dr. R. Lucile Anderson, Dr. L. G. Pooler (department of physics);
University of Michigan: Dr. T. S. Peterson;
Muhlenberg College: Mr. T. L. Koehler;
University of New Mexico: Mr. Gordon Fuller;
Oberlin College: Dr. Mildred Allen (research instructor);
Ohio State University: Dr. H. P. Thielman.

Dr. F. M. Exner, professor of geophysics at the University of Vienna, and author of *Dynamische Meteorologie*, died February 7, 1930, at the age of fifty-three.

Professor Adolf Kneser, of the University of Breslau, is dead.

Professor Ottorino Pomini, of the Milan Technical School, died March 15, 1930.

Professor G. A. Gibson, of the University of Glasgow, died April 1, 1930, at the age of seventy-one. Professor Gibson had been a member of the American Mathematical Society since 1903.

Dr. Ludwig Kotany, formerly treasurer of Washington University, and writer on economics, died May 14, 1930, at the age of sixty-nine. Dr. Kotany received his doctorate in mathematics from the University of Vienna.

Dr. T. E. McKinney, for twenty years professor of mathematics at the University of South Dakota, died April 14, 1930, at the age of sixty-six. Professor McKinney had been a member of the American Mathematical Society since 1895.

Dr. J. L. Markley, professor emeritus of mathematics at the University of Michigan, died April 20, 1930, in his seventy-first year. Professor Markley had been a member of the American Mathematical Society since 1891.

Associate Professor E. A. Pattengill, of Iowa State College, died February 10, 1930, at the age of fifty-five. Professor Pattengill was a member of the American Mathematical Society.

Dr. W. E. Story, professor emeritus of mathematics at Clark University, died April 11, 1930, at the age of seventy-nine. Professor Story had been a member of the American Mathematical Society since 1897.

Mr. R. W. Weeks, formerly vice-president and chief actuary of the New York Life Insurance Company, died April 18, 1930. Mr. Weeks had been a member of the American Mathematical Society since 1891.

ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THIS SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume.* Cross-references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

290. Dr. S. S. Cairns: *The cellular division and approximation of regular spreads.*

An i-cell (Veblen, *Analysis Situs*, The Cambridge Colloquium, Part II, pp. 73–74), ($i=1, 2, 3, \cdots, n$), which is definable by a correspondence of class C' is called *regular*. An *i-spread* is a set of points each of which has for a neighborhood on the set an i-cell or an i-cell plus one of its bounding $(i-1)$-cells. The latter sort of points constitute the boundary of the spread. If (a) the i-cells of this definition are all regular, and (b) the boundary consists of a number of $(i-1)$-circuits (Veblen, loc. cit.) made up of regular $(i-1)$-cells, then the i-spread is called *completely regular*. A *regular i-spread* is one which possesses these properties save on certain $(i-1)$-circuits made up of regular $(i-1)$-cells. It is shown in this paper that a region of euclidean 3-space bounded by a number of regular 2-spreads without boundary is a complex (Veblen, loc. cit.); and that this theorem can be extended to any number of dimensions if the bounding spreads are completely regular. It is shown also that for any regular 2-spread in euclidean 3-space there exists an approximating completely regular 2-spread such that (a) the approximating spread coincides with the given spread save at points arbitrarily near the irregularities of the latter; and (b) a continuous one-to-one correspondence can be set up between the two spreads such that the distance between any pair of corresponding points is arbitrarily small. (Received June 10, 1930.)

291. Professor D. R. Davis: *On minimizing certain types of definite integrals.*

Two types of definite integrals of the form

$$I_1=\int_{x_1}^{x_2} z^n(1+y'^2+z'^2)^m dx,\quad I_2=\int_{x_1}^{x_2}(y^2+z^2)^n(1+y'^2+z'^2)^m dx$$

are considered. In each case the expression for the difference between the value of the integral taken along an extremal arc and the value along a comparison curve is given in terms of a certain quadratic form. Hence, by means of known properties of n-ary quadratic forms sufficient conditions for the existence of an extremum for I_1, and also for I_2, are obtained. (Received May 17, 1930.)

* See pp. 1–2 and p. 45 of the January issue.

292. Dr. J. M. Feld: *The expansion of analytic functions in the form* $b_0+\sum_1^\infty b_n a_n z^n/(1-a_n z^n)$.

This paper shows that given any set of constants b_n with the condition that for every n, $|b_{n+1}| \geqq |b_n| \geqq 1$, any function $f(z)$ analytic in the neighborhood of the origin can be uniquely represented in the neighborhood of the origin as a uniformly convergent series of the form $b_0+\sum_1^\infty b_n a_n z^n/(1-a_n z^n)$. From this result it follows that if $f(0)=1$, $f(z)$ can be represented as an absolutely convergent infinite product $\prod_{n=1}^\infty (1-a_n z^n)^{-b_n/n}$ in some neighborhood of the origin. This is a generalization of a theorem by J. F. Ritt, soon to appear in the Mathematische Zeitschrift. (Received May 17, 1930.)

293. Mr. L. S. Kennison: *Linear functional equations in the composite range of a function and n independant variables.*

A systematic theory is developed from the start for the system

$$y(x)=\bar{y}(x)+\int_a^b K(x, s).\bar{y}(s)ds+\sum_{i=1}^n K_i(x)\bar{y}^i,$$

$$y^i=\bar{y}^i+\int_a^b K^i(s).\bar{y}(s).ds+\sum_{j=1}^n K_j{}^i\bar{y}^j.$$

This includes a study of the Fréchet differential of the bordered Fredholm determinant and other functionals of similar type. Direct methods are used, and explicit formulas, suitable for future applications, are given. All such transformations whose bordered Fredholm determinants do not vanish are shown to form a group. (Received May 21, 1930.)

294. Professor L. H. McFarlan: *The problem of Lagrange of the calculus of variations for which the integrand contains the coordinates of the end points.*

This paper treats the problem of determining a set of functions $y_1(x), \cdots, y_n(x)$ which give a minimum value to an integral whose integrand contains $x, y_1, \cdots, y_n, y_1', \cdots, y_n'$ and the coordinates of the end points $x_1, y_{11}, \cdots, y_{n1}, x_2, y_{12}, \cdots, y_{n2}$. The arguments of the integrand also satisfy a system of $m<n$ differential equations. A transversality condition is obtained from the first variation. From the second variation, a boundary value problem is set up in terms of which one may determine the sign of the second variation and from which one may also derive a condition analogous to the Jacobi condition in the more simple problem of the calculus of variations. (Received May 19, 1930.)

295. Professor A. D. Michal and Mr. L. S. Kennison, *Quadratic functional forms in a composite range.*

This paper develops a theory of quadratic functional forms in a function $y(x)$ and n independent variables $y^1, y^2, \cdots, y^n$. This study, in particular, includes as a special case a unified theory of the classical quadratic form theory in n variables and the recent work on quadratic functional forms. An application, of some importance, is made to functional forms in $y(x)$ alone, continuous of order one, with continuous coefficients. One of the theorems proved is that

the theory of such forms can be reduced to that of forms, continuous of order zero, in a function $\phi(x)$ and a variable Y with continuous coefficients. (Received May 23, 1930.)

296. Dr. W. E. Milne: *On the maximum absolute value of the derivative of* $e^{-x^2}P_n(x)$.

A remarkable theorem due to S. Bernstein asserts that if L is the maximum absolute value of an arbitrary polynomial $P_n(x)$ of degree n in the interval (a, b) then the maximum absolute value of the derivative $P_n'(x)$ does not exceed $nL\,[(b-x)(x-a)]^{-1/2}$ on (a, b). A related theorem for trigonometric sums states that if L is the maximum of the absolute value of a trigonometric sum of order n, then the maximum absolute value of its derivative does not exceed nL.

In this paper a similar result is obtained for the function $e^{-x^2}P_n(x)$, where $P_n(x)$ is an arbitrary polynomial of degree n. It is shown that if L is the maximum absolute value of $e^{-x^2}P_n(x)$ in the interval $-\infty < x < \infty$ then the maximum absolute value of the derivative is less than $n^{1/2}L[1.0951+O(n^{-1})]$ in the infinite interval. (Received May 16, 1930.)

297. Professor R. E. Moritz: *On doubly-depleted Fourier's series of the types* $\sum_1^\infty f(n)\cos(nx)$ *and* $\sum_1^\infty f(n)\sin(nx)$, *where* $f(n)$ *is a homogeneous function of* n.

A series of the types indicated in which n is limited to positive integers prime to a given integer P is designated as a depleted Fourier's series as distinguished from a complete Fourier's series in which no terms are missing. The series is said to be singly-depleted if P is a prime number, doubly-depleted if P consists of the product of two prime factors, multiply-depleted if P consists of the product of more than two factors. The paper deals with doubly-depleted series of the types mentioned. Multiply-depleted Fourier's series will be treated in a subsequent paper. (Received May 15, 1930.)

298. Mr. J. B. Small: *A center-surface for the ellipsoid.*

Cayley discussed the center-surfaces for an ellipsoid generated by the points P_1 and P_2 on the inward normal at distances from the surface equal to the principal radii of curvature C_1 and C_2. The author determines the surface generated by a like point P, whose distance from the surface is the mean proportional of C_1 and C_2. Various interesting singularities are found, and the surface is drawn for several shapes of ellipsoids. (Received May 24, 1930.)

299. Miss Florence Swanson: *Curved wires of minimum attraction for a given particle.*

Given a particle P and two equidistant points A, B, the author studies the shape of a fine curved wire (in the plane) through A and B which shall exert upon P the minimum attraction. The differential equation arising from Euler's condition is replaced by two others, of the first order, relating r and θ to the polar directional angle ψ as an auxiliary variable. By graphical charts and step-by-step integration a series of extremals are determined, and the curves drawn. Sufficiency tests are presented. (Received May 24, 1930.)

300. Professor J. W. Campbell: *On the determination of stringing tensions for transmission lines and cables.*

In this paper there is given a means of calculating a temperature-tension relation for any given span. The problem is treated under three cases: (1) points of support at the same level; (2) points of support at different levels; (3) composite spans which consist of several individual spans to be strung simultaneously. Hyperbolic methods are used throughout, and the details are arranged to facilitate computations. In the concluding paragraph it is shown that while the tension in a hanging cable is not uniform over the length, it is unnecessary to consider this variation in tension when finding a temperature-tension relation for the span. (Received June 20, 1930.)

301. Professor J. V. Uspensky: *Incomplete numerical functions.*

In a paper *On Gierster's Class-Number Relations* the author has given a very general identity similar to those published by Liouville. In the present paper he gives another general identity of the same kind and devises several curious formulas involving incomplete numerical functions. (Received June 20, 1930.)

302. Professor J. V. Uspensky: *Ch. Jordan's series for probability.*

In an article published in the Bulletin de la Societé Mathématique de France, Ch. Jordan gave a very remarkable series capable of representing a given infinite series of numbers and made use of this series to represent the probabilities in the Bernoullian series of trials. Ch. Jordan's development is purely formal and does not contain a method to establish the convergence of his series. In the present paper the question concerning the convergence of Ch. Jordan's series is solved under very general assumptions and moreover, the practically important question of the rapidity of the convergence of those series is elucidated. (Received June 20, 1930.)

303. Professor A. D. Michal: *Theorems concerning functional geodesic coordinates.*

The theory and application of the subject of n-dimensional geodesic coordinates of order r has been given by the author in two previous papers (Abstracts 36–3–159 and 36–3–209). The present paper is concerned with corresponding theories in an affinely connected function space. Certain applications are given in the next paper. (Received July 1, 1930.)

304. Professor A. D. Michal: *Scalar functional invariants of a continuous one-parameter family of functional vectors.*

This paper is concerned with applications of some of the theorems of the author's preceding paper, *Theorems concerning functional geodesic coordinates*, to the calculation of absolute scalar functional invariants of a continuous one-parameter family of contravariant (covariant) functional vectors

in a function space with a symmetric affine connection. In particular, the analogues of Ricci's coefficients of rotation are introduced. (Received July 1, 1930.)

305. Professor A. D. Michal: *An introduction to differential geometries of composite spaces.* Preliminary report.

This paper marks the beginnings of a theory of tensors and various types of differential geometries in the composite spaces whose points are determined by a function $y(s)$, $a \leqq s \leqq b$, and n independent variables $x_1, x_2, \cdots, x_n$. Theories of "linear connections," "covariant differentiation," and "infinitesimal parallelism" are developed. As explicit special cases of the general subject of this paper, one can cite two noteworthy instances: (1) the theory of n-dimensional affinely connected manifolds; (2) the author's studies on differential geometries of the space of the continuous functions $y(s)$ (American Journal, 1928; Proceedings of the National Academy, 1930). (Received July 1, 1930.)

306. Mr. L. S. Kennison: *Projective transformations in function space.*

An application of work on linear functional equations in a composite range is made to projective transformations in function space. All the results given by L. L. Dines (Transactions of this Society, vol. 20 (1919), p. 45) are obtained somewhat more simply. It is proved for the first time that the one-parameter family of finite transformations generated by an infinitesimal projective functional transformation is a one-parameter continuous group of projective functional transformations. (Received July 1, 1930.)

307. Professor W. L. Ayres: *On the regular points of a continuum.*

In this paper a study is made of the points of finite order (in the sense of Urysohn and Menger) in any continuum. It is shown that for $n>2$ the set of all points of order n forms a zero-dimensional set. This is a corollary of a more general result that the set of all points p such that $n \leqq$ order of $p \leqq 2n-3$ is a zero-dimensional set. It is proved that the set of points of order 2 consists of a countable number of arcs plus a zero-dimensional set, and each arc-segment is an open subset of the continuum. As it is already known that the points of order 1 are at most zero-dimensional, this completes the knowledge of the structure of the points of order n for any finite integer n. (Received May 21, 1930.)

308. Mr. K. E. Bisshopp: *Abstract defining relations for the simple group of order* 5616.

The simple group of order 5616 may be generated by an operator T of order 2 and an S of order 3. The product of these two must be of order 13, and they must satisfy one of two sets of conditions. The particular set of relations satisfied by such an S and T is also satisfied by S^{-1} and T. (Received July 7, 1930.)

309. Professor John Eiesland: *On a certain class of ruled V_{n-1}^{n-1} in S_n.* Second paper.

This paper is the continuation of one which will appear in the next issue of the Palermo Rendiconti. The singular loci of the general V_{n-1}^{n-1} in even and odd space have been enumerated. In even space it is found that, besides the *regular loci* such as $(n/2)$-fold points, $((n-2)/2)$-fold 2-flats, $\cdots$, r-fold $(n-2r)$-flats, there exist also double, triple, $\cdots$ r-fold spreads of dimension $d=n-2r+2-2^{r-1}$. These are the *accessory singular loci*. All the singular loci are of even dimension. In odd space, we have a similar result, but the singular loci are now of odd dimension. A few interesting properties of these accessory loci have been considered. The V_8^5 in S_6 and the V_6^6 in S_7 have been treated somewhat in detail. The Grassmannian coordinates have been used throughout. (Received July 5, 1930.)

310. Professor O. E. Glenn: *Preliminary report on a new method in the theory of central orbits.*

Invariant equations, which we derive from integral invariants, represent transcendental curves each having a segment in practical coincidence with a given segment of the orbit. The accuracy of the coincidence increases with a parameter. The result is an advantageous viewpoint, in problems relating to an orbit, where a segment is the primary object of study, for the reason that the transcendental curves are simpler to deal with analytically than the orbital equation itself. An instance of such a problem is the central potential which can be determined from a segment. (Received June 26, 1930.)

311. Professor J. I. Hutchinson: *Note on the number of linearly independent Dirichlet series that satisfy certain functional equations.*

In a paper published in the Transactions of this Society, volume 31, pages 324–344, which considers four different types of functional equations satisfied by ordinary Dirichlet series, the author was unable to give a rigorous determination of the number of linearly independent functions (of the form considered) in two out of the three cases that arise. The present note fills this gap by determining in a very simple manner the multiplicity of the roots of the characteristic determinant on which the solution of the problem depends. This is done by observing that the sum of these roots is equal to the sum of the coefficients occurring in the main diagonal of the determinant, and this sum can be evaluated in all cases by means of Gauss's sums. The results confirm the conjectures advanced in the paper referred to. (Received June 24, 1930.)

312. Professor M. H. Ingraham: *An elementary theorem on matrices.*

By noting the equivalence of certain matrix problems to problems in the elementary theory of numbers, a generalization to matrices of the theorem concerning the existence of a polynomial of degree $n-1$ having given values for n distinct values of the variable is found. Among other results the following

is noted. If the minimum equations of n finite square matrices m_1 to m_n with elements in a field F are relatively prime, then, for any set of n polynomials $p_1, \cdots, p_n$ in F, a polynomial f may be found such that $f(m_i)=h_i(m_i)$, $i=1, \cdots, n$. The solution is essentially unique. (Received May 5, 1930.)

313. Professor Edward Kasner: *Element transformations and isogonal series.*

An isogonal series is defined by the author to mean ∞^1 elements (x, y, p) which make a constant angle α with the curve obtained by joining the points of the elements. If $\alpha=0$ the series becomes a union or curve. For an arbitrary element transformation it is shown that there exist ∞^4 isogonal series which are converted into isogonal series. This quadruple family includes the triple family discussed in the author's previous paper (Abstract 36–5–239), namely the curves turned into isogonal series; also of course the ∞^2 curves turned into curves (American Journal, 1910). For any given value of α, the base curves of the ∞^3 series have properties analogous to those for the special case $\alpha=0$. But for a given value of the angle β of the transformed series the situation is much more complicated. (Received June 12, 1930.)

314. Dr. N. H. McCoy (National Research Fellow): *On some general commutation formulas.*

Let $x_i (i=1, 2, \cdots, n)$ be any class of elements satisfying the ordinary laws of algebra except that multiplication is non-commutative. In this paper we obtain two identities for the commutator, $fg-gf$, where f is an arbitrary defined function of the x's and g is of the form $\sum a_{i_1, i_2, \cdots, i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$. The identities are given in terms of expressions of the form $\phi x_i - x_i \phi$, where ϕ is a function of the x's. From these identities we can obtain some general commutation formulas for functions of p and q, these being subject to the relation $pq-qp=c$, which is characteristic of quantum mechanics. The formulas obtained here are in a different form from those given in a previous paper. It is found, for example, that if f and g are arbitrary polynomials in p and q, then $fg-gf=\sum c^s \sum_{i+j=s} [(-1)^i/(i!j!)] [(\partial^s g/\partial p^i \partial q^i) - (\partial^s f/\partial q^i \partial p^i)]$. Corresponding results are obtained for the case of n pairs of quantum variables satisfying similar relations. Some further applications are made to functions of three variables, α, β, γ, which satisfy the following commutation rules: $\alpha\beta-\beta\alpha=c\gamma$, $\gamma\alpha-\alpha\gamma=c\beta$, $\beta\gamma-\gamma\beta=c\alpha$. (Received July 5, 1930.)

315. Professor G. Y. Rainich: *On interlinkings.*

Gauss gave a formula for the number of interlinkings of two curves in the form of a double integral. This formula is proved in the present paper by first showing that the integral is a special case of the Kronecker-Picard integral and, therefore, its value is always an integer; it follows that a continuous deformation of the curves during which they do not acquire common points does not change the value of the integral; this permits us to reduce the general case to standard cases in which the calculation is very easy, and so to prove Gauss's formula. Next the formula is generalized for two closed subspaces of a space whose number of dimensions exceeds by one the sum of the numbers of dimensions of the subspaces. (Received July 7, 1930.)

316. Professor C. N. Reynolds: *Closed circuits upon polyhedra.*

This paper deals with the conditions under which it is possible to assign a cyclic order to the regions (or vertices) of a polyhedron such that any two consecutive elements are incident with one and only one edge of the polyhedron. An earlier result due to Mr. Whitney (Abstract 36–3–187) is extended, and the relations between this problem and the four-color problem are considered. The methods used are suggested by the theory of those maps which are irreducible with respect to the four-color problem. (Received July 7, 1930.)

American Mathematical Society

COLLOQUIUM SERIES

Published in May, 1930:

VOLUME XI. **The Theory of Approximation,** by Dunham Jackson. New York, 1930. $2.50.

EARLIER VOLUMES

VOLUME I. **Linear Systems of Curves on Algebraic Surfaces,** by H. S. White; **Forms of Non-Euclidean Space,** by F. S. Woods; **Selected Topics in the Theory of Divergent Series and of Continued Fractions,** by E. B. Van Vleck. (Boston Colloquium.) New York, 1905. $2.75.

VOLUME II. **New Haven Colloquium,** by E. H. Moore, E. J. Wilczynski, and Max Mason. Out of print.

VOLUME III. **Fundamental Existence Theorems,** by G. A. Bliss; **Differential-Geometric Aspects of Dynamics,** by Edward Kasner. (Princeton Colloquium.) New York, 1913. $2.50.

VOLUME IV. **On Invariants and the Theory of Numbers,** by L. E. Dickson; **Topics in the Theory of Functions of Several Complex Variables,** by W. F. Osgood. (Madison Colloquium.) New York, 1914. $2.50.

VOLUME V, PART I. **Functionals and their Applications. Selected Topics including Integral Equations,** by G. C. Evans. New York, 1918. $2.00. PART II. **Analysis Situs,** by Oswald Veblen, out of print. (Cambridge Colloquium.)

VOLUME VI. **The Logarithmic Potential. Discontinuous Dirichlet and Neumann Problems,** by G. C. Evans. New York, 1927. $2.00.

VOLUME VII. **Algebraic Arithmetic,** by E. T. Bell. New York, 1927. $2.50.

VOLUME VIII. **Non-Riemannian Geometry,** by L. P. Eisenhart. New York, 1927. $2.50.

VOLUME IX. **Dynamical Systems,** by G. D. Birkhoff. New York, 1927. $3.00.

VOLUME X. **Algebraic Geometry and Theta Functions,** by A. B. Coble. New York, 1929. $3.00.

Orders may be sent to the American Mathematical Society, 450 Ahnaip St., Menasha, Wis., or 501 West 116th Street, New York City; or to

The Open Court Publishing Co., 339 East Chicago Ave., Chicago, Ill.
Bowes and Bowes, 1 Trinity Street, Cambridge, England.
Hirschwaldsche Buchhandlung, Unter den Linden 68, Berlin, Germany.

A special discount of 50 cents per volume is allowed to members of the American Mathematical Society, on all the books listed above when orders are sent by the member directly to the office of the Society in New York City.

Mémorial des Sciences Mathématiques

Edited by HENRI VILLAT

Under the patronage of la Société mathématique de France, l'Académie des Sciences de Paris, and the scientific academies of Servia, Belgium, Bucarest, Poland, Ukraine, Spain, Prague, Rome (dei Lincei), and Sweden. See this BULLETIN, vol. 31, Nos. 5-6, p. 281.

In addition to twenty-five monographs that appeared before 1928, the following have recently been published.

26. B. GAMBIER, *Déformation des surfaces étudiées du point de vue infinitésimal.* 1928.
27. P. APPELL, *Le problème géométrique des déblais et remblais.* 1928.
28. E. COTTON, *Approximations successives et équations différentielles.* 1928.
29. C. GUICHARD, *Les courbes de l'espace à n dimensions.* 1928.
30. L. ZORETTI, *Les principes de la mécanique classique.* 1928.
31. B. GAMBIER, *Applicabilité des surfaces étudiée du point de vue fini.* 1928.
32. CH. RIQUIER, *La méthode des fonctions majorantes et les systèmes d'équations aux dérivées partielles.* 1928.
33. A. BUHL, *Aperçus modernes sur la théorie des groupes continus et finis.* 1928.
34. H. VERGNE, *Ondes liquides de gravité.* 1928.
35. LÉON LECORNU, *Théorie mathématique de l'élasticité.* 1929.
36. PAUL APPELL, *Sur la décomposition d'une fonction méromorphe en éléments simples.* 1929.
37. G. CERF, *Transformations de contact et problème de Pfaff.* 1929.
38. G. VALIRON, *Familles normales et quasi-normales de fonctions méromorphes.* 1929.
39. T. NAGELL, *L'analyse indéterminée de degré supérieur.* 1929.
40. S. LEFSCHETZ, *Géométrie sur les surfaces et les variétés algébriques.* 1929.

Many others have appeared or are in preparation.

For all information, address

Gauthier-Villars et Cie., 55, Quai des Grands-Augustins, Paris

BOOKS WANTED

Under this head will be printed very special needs of the Society itself, and in so far as possible the needs of other non-profit-making bodies, particularly Departments of Mathematics in Colleges and Universities; but needs that can be supplied through obvious commercial channels should not appear here.

Wanted by the Department of Mathematics, Wells College:

Bolza, Oskar: Lectures on the calculus of variations, Chicago U. Press, 1904.

Collectanea Mathematica in memoriam Dominici Chelini, Hoepli, 1881.

Memorie della reale accademia delle scienze di Torino, Ser. 2, Vol. 52, 1903.

Annali di Matematica: Ser. 1; Ser. 2, Vols. 1-4, 22, 24-26; Ser. 3, Vols. 14, 16.

Rendiconti del Circ. Matematico di Palermo, Vols. 35, 38, 40.

Carmichael-Smith

PLANE AND SPHERICAL TRIGONOMETRY

A new course by R. D. Carmichael, Head of the Mathematics Department, University of Illinois, and E. R. Smith, Professor of Mathematics, Iowa State College. The early chapters emphasize the practical aspects of trigonometry, thus stimulating interest and paving the way for the more abstract and more analytical parts of the subject which follow. The needs of the student are kept carefully in mind in the exposition and development. Catalogue price $1.60.

Boston Chicago **GINN AND COMPANY** Atlanta Dallas Columbus New York San Francisco

The Carus Mathematical Monographs

The fourth volume in the series of Carus Mathematical Monographs was published in April, 1930. The title is "Projective Geometry"; the author is Professor J. W. Young of Dartmouth College.

Copies of this and the preceding monographs may be obtained by members of the Mathematical Association of America at cost through Secretary W. D. Cairns, Oberlin, Ohio. All other orders should be sent to the Open Court Publishing Company, Chicago, Ill. Price $1.25 each to members, $2.00 to others.

OFFICIAL COMMUNICATIONS

Meetings of the Society have been fixed at the following times and places:

PROVIDENCE, RHODE ISLAND, SUMMER MEETING AND COLLOQUIUM, September 9-12, 1930.

Abstracts must be in the hands of the Secretary of the Society, Professor R. G. D. Richardson, 501 West 116th Street, New York City, not later than August 11. In order to be printed in the Bulletin in advance of the meeting, abstracts must be in the hands of the Secretary not later than July 7. Professor Solomon Lefschetz will deliver a series of colloquium lectures on Topology. Professors T. H. Hildebrandt and J. D. Tamarkin will deliver addresses on topics in the general theory of linear operations and integral equations.

NEW YORK CITY, October 25, 1930.

Abstracts must be in the hands of the Secretary of the Society, Professor R. G. D. Richardson, 501 West 116th Street, New York City, not later than October 4. In order to be printed in the Bulletin in advance of the meeting, abstracts must be in the hands of the Secretary not later than September 7. By invitation of the program committee, Professor M. H. Stone will deliver an address on group representation in Hilbert space.

COLUMBIA, MISSOURI, November 28-29, 1930.

Abstracts must be in the hands of Associate Secretary M. H. Ingraham, North Hall, Madison, Wisconsin, not later than November 6. All abstracts received by that date will appear in the Bulletin in advance of the meeting. On Saturday morning, Professor C. N. Moore will deliver an address by invitation of the program committee, on types of series and types of summability; and Professor R. E. Langer will deliver an address on the zeros of exponential sums and of certain related functions.

UNIVERSITY OF CALIFORNIA AT LOS ANGELES, November 29, 1930.

Abstracts must be in the hands of Associate Secretary T. M. Putnam, Berkeley, California, not later than November 1. All abstracts received by that date will appear in the Bulletin in advance of the meeting.

CLEVELAND, OHIO, ANNUAL MEETING, December 29, 1930 to January 1, 1931.

Abstracts must be in the hands of Associate Secretary M. H. Ingraham, North Hall, Madison, Wisconsin, not later than December 1. In order to be printed in the Bulletin in advance of the meeting, abstracts must be in the hands of the Associate Secretary not later than November 7.

R. G. D. RICHARDSON, *Secretary of the Society.*

Articles for insertion in the BULLETIN should be addressed to E. R. HEDRICK, Editor of the BULLETIN, University of California at Los Angeles. Reviews should be sent to W. R. LONGLEY, Yale University, New Haven, Conn. Notes should be sent to H. W. KUHN, Ohio State University, Columbus, Ohio.

Subscriptions to the BULLETIN, orders for back numbers, and inquiries in regard to non-delivery of current numbers should be addressed to the American Mathematical Society, 450-458 Ahnaip St., Menasha, Wis., or 501 West 116th Street, New York.

The initiation fees ($5.00) and the annual dues ($6.00) of members of the Society are payable to the Treasurer of the Society, G. W. MULLINS, 501 West 116th Street, New York.

WHOLE NUMBER 370

CONTENTS

	PAGE
The June Meeting in Oregon. By T. M. PUTNAM	449
Coble on Algebraic Geometry and Theta Functions. By OSCAR ZARISKI	452
Dickson's Theory of Numbers. By E. T. BELL	455
The New Mechanics. By R. D. CARMICHAEL	459
Lebesgue on Integration. By T. H. HILDEBRANDT	463
Shorter Notices	469
Notes	475
Abstracts of Papers	481

For official Communications and Notices, see the inside of the back cover.